Homework 3 Solution

1. We have that

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) = \int_0^\infty e^{ty} f_Y(y)\,dy \\
&= \int_0^\infty e^{ty} y e^{-y}\,dy = \int_0^\infty y e^{-(1-t)y}\,dy \\
&= \int_0^\infty y^{2-1} e^{-(1-t)y}\,dy = \frac{\Gamma(2)}{(1-t)^2}\int_0^\infty \frac{(1-t)^2}{\Gamma(2)} y^{2-1} e^{-(1-t)y}\,dy \\
&= \frac{\Gamma(2)}{(1-t)^2}\int_0^\infty \frac{1}{\Gamma(2)[1/(1-t)]^2} y^{2-1} e^{-y/[1/(1-t)]}\,dy \\
&= \frac{\Gamma(2)}{(1-t)^2}\cdot 1 \\
&= \frac{1}{(1-t)^2} = (1-t)^{-2}
\end{aligned}$$

as desired. We realize that $Y \sim \text{Gamma}(2, 1)$.

2. Here,

$$\begin{aligned}
M_Y(t) &= E(e^{tY}) \\
&= \int_{-1}^0 e^{ty}(1+y)\,dy + \int_0^1 e^{ty}(1-y)\,dy \\
&= \int_{-1}^0 e^{ty}\,dy + \int_{-1}^0 ye^{ty}\,dy + \int_0^1 e^{ty}\,dy - \int_0^1 ye^{ty}\,dy \\
&= \int_{-1}^1 e^{ty}\,dy + \int_{-1}^0 ye^{ty}\,dy - \int_0^1 ye^{ty}\,dy \\
&= \frac{1}{t}e^{ty}\Big|_{-1}^1 + \frac{x}{t}e^{ty}\Big|_{-1}^0 - \int_{-1}^0 \frac{1}{t}e^{ty}\,dy - \frac{x}{t}e^{ty}\Big|_0^1 + \int_0^1 \frac{1}{t}e^{ty}\,dy \\
&= \frac{1}{t}e^t - \frac{1}{t}e^{-t} + \frac{1}{t}e^{-t} - \frac{1}{t^2} + \frac{1}{t^2}e^{-t} - \frac{1}{t}e^t + \frac{1}{t^2}e^t - \frac{1}{t^2} \\
&= \frac{1}{t^2}(e^t + e^{-t} - 2)
\end{aligned}$$

3. Since

$$\begin{aligned}
M_Y'(t) &= \frac{d}{dt}M_Y(t) = \frac{d}{dt}e^{t^2/2} = te^{t^2/2} \\
M_Y''(t) &= \frac{d}{dt}M_Y'(t) = \frac{d}{dt}te^{t^2/2} = t^2e^{t^2/2} + e^{t^2/2} = (t^2+1)e^{t^2/2} \\
M_Y^{(3)}(t) &= \frac{d}{dt}M_Y''(t) = \frac{d}{dt}(t^2+1)e^{t^2/2} = (t^2+1)te^{t^2/2} + 2te^{t^2/2}
\end{aligned}$$

we have that

$$E(Y^3) = M_Y^{(3)}(0) = \frac{d^3}{dt^3} M_Y(t)\Big|_{t=0} = (t^2+1)te^{t^2/2} + 2te^{t^2/2}\Big|_{t=0} = 0$$

(It follows that $E(Y^3) = 0$ for $Y \sim N(0,1)$).

4. If $Y \sim$ Binomial(n,p), then $M_Y(t) = [(1-p)+pe^t]^n$. If we let $U = n - Y$, it follows that

$$M_U(t) = M_{n-Y}(t) = e^{nt}M_Y(-t) = e^{nt}[(1-p)+pe^{-t}]^n = [e^t((1-p)+pe^{-t})]^n = [p+(1-p)e^t]^n$$

which is the mgf of Binomial$(n, 1-p)$. Hence, we conclude that $U \sim$ Binomial$(n, 1-p)$.

5. If $Y \sim$ Gamma$(n/2, m)$, then the mgf of Y is given by $M_Y(t) = (1-mt)^{-n/2}$. Thus, the mgf of $W = 2Y/m$ will then be

$$\begin{aligned} M_W(t) &= M_{2Y/m}(t) = E(e^{t(2Y/m)}) = E(e^{(2t/m)Y}) = M_Y(2t/m) = (1-m(2t/m))^{-n/2} \\ &= (1-2t)^{-n/2} \end{aligned}$$

which is the mgf of Gamma$(n/2, 2)$ or $\chi^2(n)$. Hence, $W = 2Y/m \sim \chi^2(n)$.

6. We recognize

$$M_Y(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$$

as the mgf of $Y \sim$ Binomial(5,3/4). Hence

$$\begin{aligned} P(Y \le 2) &= \sum_{y=0}^{2} \binom{5}{y} \left(\frac{3}{4}\right)^y \left(\frac{1}{4}\right)^{5-y} \\ &= (1/4)^5 + 5(3/4)(1/4)^4 + 10(3/4)^2(1/4)^3 \\ &= 0.104 \end{aligned}$$

7. If $Y_1 \sim$ Poisson(λ_1) and $Y_2 \sim$ Poisson(λ_2) are independent, then the mgfs of Y_1 and Y_2 are

$$M_{Y_1}(t) = \exp[\lambda_1(e^t-1)] \quad \text{and} \quad M_{Y_2}(t) = \exp[\lambda_2(e^t-1)]$$

respectively. Since Y_1 and Y_2 are independent, the mfg of $U = Y_1 + Y_2$ is given by

$$\begin{aligned} M_U(t) &= M_{Y_1+Y_2}(t) = M_{Y_1}(t)M_{Y_2}(t) = \exp[\lambda_1(e^t-1)]\exp[\lambda_2(e^t-1)] \\ &= \exp[\lambda_1(e^t-1)+\lambda_2(e^t-1)] = \exp[(\lambda_1+\lambda_2)(e^t-1)] \end{aligned}$$

which is the mgf of Poisson$(\lambda_1+\lambda_2)$. Hence, $U = Y_1 + Y_2 \sim$ Poisson$(\lambda_1+\lambda_2)$.

8. Since

$$M_{Y_1}(t) = \left(\frac{3}{4}e^t + 1 - \frac{3}{4}\right)^5 = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$$

and

$$M_{Y_2}(t) = \exp[2(e^t-1)] = e^{2(e^t-1)}$$

and since $M_{Y_1+Y_2}(t) = M_{Y_1}(t)M_{Y_2}(t)$ if Y_1 and Y_2 are independent, the joint mgf of Y_1 and Y_2 is

$$M_{Y_1+Y_2}(t) = M_{Y_1}(t)M_{Y_2}(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5 e^{2(e^t-1)}$$

9. From Example 6.11, we see that $Y_1^2 \sim \chi^2(1)$ and $Y_2^2 \sim \chi^2(1)$, with $M_{Y_1^2}(t) = (1-2t)^{-1/2}$ and $M_{Y_2^2}(t) = (1-2t)^{-1/2}$. Since Y_1 and Y_2 are independent, then Y_1^2 and Y_2^2 are independent, so that the mfg of $U = Y_1^2 + Y_2^2$ can be computed by
$$\begin{aligned} M_U(t) &= M_{Y_1^2+Y_2^2}(t) = M_{Y_1^2}(t) M_{Y_2^2}(t) = (1-2t)^{-1/2}(1-2t)^{-1/2} \\ &= (1-2t)^{-1} = (1-2t)^{-2/2} \end{aligned}$$
which is the mgf of Exponential(2) = Gamma(1, 2) = $\chi^2(2)$. Hence, $U = Y_1^2 + Y_2^2 \sim \chi^2(2)$.

10. (a)
$$M_Y(t) = E(e^{tY}) = \sum_y e^{ty} P(Y=y) = \sum_{y=0}^{1} e^{ty} p^y (1-p)^{1-y} = 1 - p + e^t p$$
(b) Since we can write $M_Y(t) = 1 - p + e^t p = e^t p + (1-p)$, we obtain
$$\begin{aligned} M_{Y_1+\cdots+Y_n}(t) &= M_{Y_1}(t) \times \cdots \times M_{Y_n}(t) = [e^t p + (1-p)] \times \cdots \times [e^t p + (1-p)] \\ &= [e^t p + (1-p)]^n \end{aligned}$$
which is the mgf of Binomial(n, p). Hence, $Y_1 + \cdots + Y_n \sim$ Binomial(n, p).

11. Since
$$M_Y(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$
we obtain
$$\begin{aligned} M_{\overline{Y}}(t) &= E(e^{t\overline{Y}}) = E(e^{t(Y_1+\cdots+Y_n)/n}) = E(e^{(t/n)Y_1+\cdots+(t/n)Y_n} = E(e^{(t/n)Y_1}) \times \cdots \times E(e^{(t/n)Y_n}) \\ &= M_{Y_1}(t/n) \times \cdots \times M_{Y_n}(t/n) = [M_{Y_1}(t/n)]^n \quad \text{(since } Y_1, \ldots, Y_n \text{ are iid)} \\ &= \left[\exp\left(\mu(t/n) + \frac{\sigma^2 (t/n)^2}{2}\right)\right]^n = \left[\exp\left(\frac{\mu t}{n} + \frac{\sigma^2 t^2}{2n^2}\right)\right]^n = \exp\left(\mu t + \frac{\sigma^2 t^2}{2n}\right) \\ &= \exp\left(\mu t + \frac{(\sigma^2/n) t^2}{2}\right) \end{aligned}$$
which is the mgf of $N(\mu, \sigma^2/n)$. Hence, $\overline{Y} \sim N(\mu, \sigma^2/n)$.
(Alternatively, you can also use Theorem 6.3 with $\mu_i = \mu$, $\sigma_i^2 = \sigma^2$, $a_i = 1/n$).

12. **(5880*)** We have that
$$M_{Y_1}(t) = (1-\beta t)^{-\alpha_1}, \quad M_{Y_2}(t) = (1-\beta t)^{-\alpha_2}, \quad M_{Y_3}(t) = (1-\beta t)^{-\alpha_3}$$
Hence,
$$\begin{aligned} M_U(t) &= M_{c_1Y_1 + c_2 Y 2 + c_3 Y_3}(t) = M_{Y_1}(c_1 t) M_{Y_2}(c_2 t) M_{Y_3}(c_3 t) \\ &= (1-\beta c_1 t)^{-\alpha_1}(1-\beta c_2 t)^{-\alpha_2}(1-\beta c_3 t)^{-\alpha_3} \end{aligned}$$
Now, if we want $M_U(t)$ to have the mgf of a gamma distribution, then we must have that $c = c_1 = c_2 = c_3$, so that
$$\begin{aligned} M_U(t) &= (1-\beta c_1 t)^{-\alpha_1}(1-\beta c_2 t)^{-\alpha_2}(1-\beta c_3 t)^{-\alpha_3} = (1-\beta c t)^{-\alpha_1}(1-\beta c t)^{-\alpha_2}(1-\beta c t)^{-\alpha_3} \\ &= (1-c\beta t)^{-(\alpha_1+\alpha_2+\alpha_3)} \end{aligned}$$
which is the mgf of Gamma$(\alpha_1 + \alpha_2 + \alpha_3, c\beta)$, so that $U \sim$ Gamma$(\alpha_1 + \alpha_2 + \alpha_3, c\beta)$.