

Homework 4 Solution

1. We can easily see that

$$\bar{X} \sim N(\mu_X, \sigma_X^2/m) \quad \text{and} \quad \bar{Y} \sim N(\mu_Y, \sigma_Y^2/n)$$

Hence,

(a)

$$E(\bar{X} - \bar{Y}) = \mu_X - \mu_Y$$

(b)

$$\text{Var}(\bar{X} - \bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y}) = \frac{\sigma_X^2}{m} + \frac{\sigma_Y^2}{n}$$

using the independence between X_i and Y_i .

2. Since $U = (n-1)S^2/\sigma^2 \sim \chi^2(n-1)$, we have that

$$M_U(t) = (1-2t)^{-(n-1)/2}$$

Then

$$E\left(\frac{(n-1)S^2}{\sigma^2}\right) = E(U) = \frac{d}{dt}M_U(t)\Big|_{t=0} = +\frac{(n-1)}{2}(1-2t)^{-(n-1)/2-1} \cdot 2\Big|_{t=0} = (n-1)$$

and therefore

$$E(S^2) = \frac{(n-1)}{(n-1)/\sigma^2} = \sigma^2$$

3. The mgf of S^2 is given by

$$M_{S^2}(t) = E(e^{tS^2}) = E\left(\exp\left(\frac{(n-1)S^2}{\sigma^2} \frac{t\sigma^2}{(n-1)}\right)\right) = E\left(\exp\left(U \frac{t\sigma^2}{(n-1)}\right)\right)$$

where $U = (n-1)S^2/\sigma^2 \sim \chi^2(n-1)$ and $M_U(t) = (1-2t)^{-(n-1)/2}$ as in Problem 2. Then

$$M_{S^2}(t) = E\left(\exp\left(\frac{t\sigma^2}{(n-1)}U\right)\right) = M_U\left(\frac{t\sigma^2}{(n-1)}\right) = \left(1 - \frac{2t\sigma^2}{(n-1)}\right)^{-(n-1)/2}$$

and hence S^2 has a gamma distribution with $\alpha = (n-1)/2$ and $\beta = 2\sigma^2/(n-1)$
(Note here again that, by a property of gamma distribution, we have that

$$E(S^2) = \alpha\beta = \frac{(n-1)}{2} \cdot \frac{2\sigma^2}{(n-1)} = \sigma^2$$

giving us yet another proof of $E(S^2) = \sigma^2$).

4. First

$$\sum_{i=1}^n [(Y_i - \bar{Y})(\bar{Y} - \mu)] = (\bar{Y} - \mu) \sum_{i=1}^n (Y_i - \bar{Y})$$

But since

$$\sum_{i=1}^n (Y_i - \bar{Y}) = \sum_{i=1}^n Y_i - \sum_{i=1}^n \bar{Y} = \sum_{i=1}^n Y_i - n \left[\frac{1}{n} \sum_{i=1}^n Y_i \right] = \sum_{i=1}^n Y_i - \sum_{i=1}^n Y_i = 0$$

it follows immediately that

$$\sum_{i=1}^n [(Y_i - \bar{Y})(\bar{Y} - \mu)] = (\bar{Y} - \mu) \sum_{i=1}^n (Y_i - \bar{Y}) = (\bar{Y} - \mu) \cdot 0 = 0$$

(This proves that $\sum_{i=1}^n (Y_i - \mu)^2 = \sum_{i=1}^n (Y_i - \bar{Y})^2 + n(\bar{Y} - \mu)^2 + 0$).

5. Note first that (see Problem 4)

$$0 = \sum_{i=1}^n (Y_i - \bar{Y}) = (Y_1 - \bar{Y}) + \sum_{i=2}^n (Y_i - \bar{Y})$$

so that

$$-(Y_1 - \bar{Y}) = \sum_{i=2}^n (Y_i - \bar{Y})$$

By squaring both sides, we obtain

$$(Y_1 - \bar{Y})^2 = \left[\sum_{i=2}^n (Y_i - \bar{Y}) \right]^2$$

6. Using the hint,

$$\begin{aligned} \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (Y_i - Y_j)^2 &= \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (Y_i - \bar{Y} + \bar{Y} - Y_j)^2 \\ &= \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (Y_i - \bar{Y})^2 + \frac{1}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (Y_j - \bar{Y})^2 \\ &\quad + \frac{2}{2n(n-1)} \sum_{i=1}^n \sum_{j=1}^n (Y_i - \bar{Y})(Y_j - \bar{Y}) \\ &= \frac{1}{2n(n-1)} \sum_{i=1}^n n(Y_i - \bar{Y})^2 + \frac{1}{2n(n-1)} \sum_{j=1}^n n(Y_j - \bar{Y})^2 + 0 \\ &= \frac{n}{2n(n-1)} \sum_{i=1}^n (Y_i - \bar{Y})^2 + \frac{n}{2n(n-1)} \sum_{i=1}^n (Y_i - \bar{Y})^2 \\ &= \frac{2n}{2n(n-1)} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{1}{(n-1)} \sum_{i=1}^n (Y_i - \bar{Y})^2 \\ &= S^2 \end{aligned}$$

7. If Y has a t -distribution with n d.f., then it must have the form

$$Y = \frac{X_1}{\sqrt{X_2/n}}$$

where $X_1 \sim N(0, 1)$ and $X_2 \sim \chi^2(n)$ are independent. Now, by squaring, $Y^2 = X_1^2/(X_2/n)$ where $X_1^2 \sim \chi^2(1)$ and is independent of $X_2 \sim \chi^2(n)$. Therefore,

$$Y^2 = \frac{X_1^2}{X_2/n} = \frac{X_1^2/1}{X_2/n} \sim F(1, n)$$

8. If $Y \sim F(m, n)$, then Y must have the form

$$Y = \frac{X_1/m}{X_2/n}$$

where $X_1 \sim \chi^2(m)$ and $X_2 \sim \chi^2(n)$ are independent. It follows that

$$\frac{1}{Y} = \frac{X_2/n}{X_1/m} \sim F(n, m)$$

9. If Y has an exponential distribution with mean θ , then the pdf of Y is

$$f_Y(y) = \frac{1}{\theta} e^{-y/\theta}, \quad y > 0, \quad \theta > 0$$

If $U = 2Y/\theta$, then $Y = \theta U/2$ and $dy/du = \theta/2$. Hence,

$$f_U(u) = f_Y(y) \left| \frac{dy}{du} \right| = \frac{1}{\theta} e^{-y/\theta} \left| \frac{\theta}{2} \right| = \frac{1}{2} e^{-u/2}, \quad u > 0$$

Since the pdf of $\chi^2(r)$ is

$$f_X(x) = \frac{1}{\Gamma(r/2) 2^{r/2}} x^{r/2-1} e^{-x/2}$$

then the pdf of $\chi^2(2)$ is $f_X(x) = 1/2 e^{-x/2}$ which coincides with the pdf $f_U(u)$, and hence U has a χ^2 distribution with 2 df. Now, if X_1 and X_2 are independent exponential distributions with mean θ , then let $U_1 = 2X_1/\theta$ and $U_2 = 2X_2/\theta$. It follows that U_1 and U_2 are independent, and each has a χ^2 distribution with 2 df, so that $(U_1/2)/(U_2/2) \sim F(2, 2)$. Finally, we see that

$$\frac{U_1/2}{U_2/2} = \frac{(2X_1/\theta)/2}{(2X_2/\theta)/2} = \frac{X_1}{X_2} \sim F(2, 2)$$

10. Since $E(S^2) = \sigma^2$, it follows by Chebyshev inequality that

$$P(|S^2 - \sigma^2| > \epsilon) = P(|S^2 - E(S^2)| > \epsilon) \leq \frac{E[|S^2 - E(S^2)|^2]}{\epsilon^2} = \frac{\text{Var}(S^2)}{\epsilon^2} \rightarrow 0$$

as $n \rightarrow \infty$, since we assumed that $\text{Var}(S^2) \rightarrow 0$ and that $\epsilon > 0$. Thus, $S_n^2 \rightarrow \sigma^2$ in probability.

11. (a) Note first that $P(X_i = x) = p^x(1-p)^{1-x}$, $x = 0, 1$. For any integer k , we have that

$$E(X_i^k) = \sum_x x^k P(X_i = x) = \sum_{x=0}^1 x^k p^x (1-p)^{1-x} = p$$

- (b) Since

$$E(Y_n) = E\left(\frac{1}{n} \sum_{i=1}^n X_i^2\right) = \frac{1}{n} \sum_{i=1}^n E(X_i^2)$$

and $E(X_i^2) = p$ from part (a), it follows that

$$E(Y_n) = \frac{1}{n} \sum_{i=1}^n E(X_i^2) = \frac{1}{n} \sum_{i=1}^n p = p$$

(You can also solve this problem by using the fact that $E(X_i^2) = \text{Var}(X_i) + [E(X_i)]^2 = p(1-p) + p^2 = p$).

- (c) First, $\text{Var}(X_i^2) = E(X_i^4) - [E(X_i^2)]^2$, and we know that $E(X_i^4) = p$ and $E(X_i^2) = p$ from part (a). Thus,

$$\text{Var}(X_i^2) = E(X_i^4) - [E(X_i^2)]^2 = p - p^2 = p(1-p)$$

Now, use the fact that $X_1, \dots, X_n$ are iid,

$$\text{Var}(Y_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i^2\right) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i^2) = \frac{1}{n^2} \sum_{i=1}^n p(1-p) = \frac{p(1-p)}{n}$$

Therefore,

$$\text{Var}(Y_n) = \frac{p(1-p)}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty$$

and thus for any $\epsilon > 0$,

$$P(|Y_n - p| > \epsilon) = P(|Y_n - E(Y_n)| > \epsilon) \leq \frac{E[|Y_n - E(Y_n)|^2]}{\epsilon^2} = \frac{\text{Var}(Y_n)}{\epsilon^2} \rightarrow 0$$

as $n \rightarrow \infty$, and therefore, $Y_n \rightarrow p$ in probability.

12. (a) If $U = (Y - \lambda)/\sqrt{\lambda}$, then

$$\begin{aligned} M_U(t) &= E[e^{tU}] = E[e^{t(Y-\lambda)/\sqrt{\lambda}}] = E[e^{(t/\sqrt{\lambda})Y} e^{-t\sqrt{\lambda}}] = e^{-t\sqrt{\lambda}} E[e^{(t/\sqrt{\lambda})Y}] \\ &= e^{-t\sqrt{\lambda}} M_Y(t/\sqrt{\lambda}) = e^{-t\sqrt{\lambda}} \exp(\lambda e^{t/\sqrt{\lambda}} - \lambda) = \exp(-t\sqrt{\lambda}) \exp(\lambda e^{t/\sqrt{\lambda}} - \lambda) \\ &= \exp(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda) \end{aligned}$$

- (b) First, using the hint,

$$e^{t/\sqrt{\lambda}} = 1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3!\lambda^{3/2}} + \dots$$

so that

$$\begin{aligned}
M_U(t) &= \exp(\lambda e^{t/\sqrt{\lambda}} - t\sqrt{\lambda} - \lambda) \\
&= \exp\left(\lambda \left[1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3!\lambda^{3/2}} + \cdots\right] - t\sqrt{\lambda} - \lambda\right) \\
&= \exp\left(\left[\lambda + t\sqrt{\lambda} + \frac{t^2}{2} + \frac{t^3}{3!\sqrt{\lambda}} + \cdots\right] - t\sqrt{\lambda} - \lambda\right) \\
&= \exp\left(\frac{t^2}{2} + \frac{t^3}{3!\sqrt{\lambda}} + \cdots\right)
\end{aligned}$$

Now, as $\lambda \rightarrow \infty$,

$$M_U(t) \rightarrow \exp\left(\frac{t^2}{2}\right)$$

because all the $\frac{t^3}{3!\sqrt{\lambda}} + \cdots$ terms go to zero, proving the result.

13. This problem can be solved in many ways, but it is easiest (or most instructive) to solve

$$\begin{aligned}
P(\min(Y_1, Y_2, Y_3, Y_4, Y_5) > a) &= P(Y_1 > a, Y_2 > a, Y_3 > a, Y_4 > a, Y_5 > a) \\
&= P(Y_1 > a)P(Y_2 > a)P(Y_3 > a)P(Y_4 > a)P(Y_5 > a) \\
&= [P(Y_1 > a)]^5 = [e^{-a/\beta}]^5 \\
&= e^{-5a/\beta}
\end{aligned}$$

14. First, the cdf of Y is given by

$$F_Y(y) = \int_{\theta}^y f_Y(t) dt = \int_{\theta}^y e^{-(t-\theta)} dt = 1 - e^{-(y-\theta)}, \quad y > \theta$$

so that

$$f_{Y_{(1)}}(y) = n[1 - F_Y(y)]^{n-1} f_Y(y) = n[1 - (1 - e^{-(y-\theta)})]^{n-1} e^{-(y-\theta)} = n[e^{-(y-\theta)}]^n = ne^{-n(y-\theta)}$$

for $y > \theta$.

(This problem can also be solved by a technique similar to the previous problem).

15. First, the pdf of Y is $f_Y(y) = 1/\theta$ and the cdf of Y is $F_Y(y) = y/\theta$, for $0 < y < \theta$, so that

$$\begin{aligned}
f_{Y_{(k)}}(y) &= \frac{n!}{(k-1)!(n-k)!} [F_Y(y)]^{k-1} [1 - F_Y(y)]^{n-k} f_Y(y) \\
&= \frac{n!}{(k-1)!(n-k)!} \left[\frac{y}{\theta}\right]^{k-1} \left[1 - \frac{y}{\theta}\right]^{n-k} \left(\frac{1}{\theta}\right), \quad 0 < y < \theta
\end{aligned}$$

Now, if $\theta = 1$, then

$$\begin{aligned}
f_{Y_{(k)}}(y) &= \frac{n!}{(k-1)!(n-k)!} y^{k-1} (1-y)^{n-k} = \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} y^{k-1} (1-y)^{n-k} \\
&= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} y^{k-1} (1-y)^{n-k+1-1}, \quad 0 < y < 1
\end{aligned}$$

and this is a pdf of beta distribution with $\alpha = k$ and $\beta = n-k+1$, i.e., $Y_{(k)} \sim \text{Beta}(k, n-k+1)$.

16. **(5880*)** If $Y_1, Y_2, \dots, Y_n \sim \text{iid } N(\mu, \sigma^2)$, then first note that the joint pdf of $(Y_1, \dots, Y_n)$ is given by

$$\begin{aligned} f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) &= f_{Y_1}(y_1) \cdots f_{Y_n}(y_n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_1 - \mu)^2}{2\sigma^2}} \times \cdots \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y_n - \mu)^2}{2\sigma^2}} \\ &= \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 \right] \end{aligned}$$

- (a) If $U_1 = \bar{Y}$, $U_2 = Y_2 - \bar{Y}$, $U_3 = Y_3 - \bar{Y}$, $\dots$, $U_n = Y_n - \bar{Y}$, then the inverse transformations are given by (work from the bottom)

$$\begin{aligned} Y_1 &= n\bar{Y} - Y_2 - Y_3 - \cdots - Y_n = U_1 - U_2 - U_3 - \cdots - U_n \\ Y_2 &= U_2 + \bar{Y} = U_2 + U_1 \\ Y_3 &= U_3 + \bar{Y} = U_3 + U_1 \\ &\vdots \\ Y_n &= U_n + \bar{Y} = U_n + U_1 \end{aligned}$$

It follows that the Jacobian is

$$J = \begin{vmatrix} \frac{dy_1}{du_1} & \frac{dy_1}{du_2} & \frac{dy_1}{du_3} & \cdots & \frac{dy_1}{du_n} \\ \frac{dy_2}{du_1} & \frac{dy_2}{du_2} & \frac{dy_2}{du_3} & \cdots & \frac{dy_2}{du_n} \\ \frac{dy_3}{du_1} & \frac{dy_3}{du_2} & \frac{dy_3}{du_3} & \cdots & \frac{dy_3}{du_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{dy_n}{du_1} & \frac{dy_n}{du_2} & \frac{dy_n}{du_3} & \cdots & \frac{dy_n}{du_n} \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = n$$

- (b) Noting from part (a) and from Problem 5 that $(Y_1 - \bar{Y})^2 = [\sum_{i=2}^n (Y_i - \bar{Y})]^2 = [\sum_{i=2}^n U_i]^2$ and $(Y_i - \bar{Y}) = U_i$ for $i = 2, 3, \dots, n$ and $\bar{Y} = U_1$, we arrive at

$$\begin{aligned} f_{U_1, \dots, U_n}(u_1, \dots, u_n) &= f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) |J| = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \sum_{i=1}^n (y_i - \mu)^2 \right] |n| \\ &= n \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \sum_{i=1}^n ((y_i - \bar{y})^2 + (\bar{y} - \mu)^2) \right] \\ &= n \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \sum_{i=1}^n (y_i - \bar{y})^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (\bar{y} - \mu)^2 \right] \\ &= n \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \left((y_1 - \bar{y})^2 + \sum_{i=2}^n (y_i - \bar{y})^2 \right) - \frac{1}{2\sigma^2} n(\bar{y} - \mu)^2 \right] \\ &= n \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \left(\left[\sum_{i=2}^n u_i \right]^2 + \sum_{i=2}^n u_i^2 \right) - \frac{n}{2\sigma^2} (u_1 - \mu)^2 \right] \\ &= n \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left[\frac{-1}{2\sigma^2} \left(\left[\sum_{i=2}^n u_i \right]^2 + \sum_{i=2}^n u_i^2 \right) \right] \exp \left[\frac{-n}{2\sigma^2} (u_1 - \mu)^2 \right] \end{aligned}$$

Since we can write $f_{U_1, \dots, U_n}(u_1, \dots, u_n) = g(u_2, u_3, \dots, u_n)h(u_1)$, it follows that any function of $U_1 = \bar{Y}$ and any function of $(U_2, U_3, \dots, U_n)$ are independent.

(c) By inspection in part (b), notice that $\sum_{i=1}^n (y_i - \bar{y})^2 = \left(\left[\sum_{i=2}^n u_i \right]^2 + \sum_{i=2}^n u_i^2 \right)$, so that

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 = \frac{1}{n-1} \left(\left[\sum_{i=2}^n U_i \right]^2 + \sum_{i=2}^n U_i^2 \right)$$

Hence, S^2 is a function of $(U_2, U_3, \dots, U_n)$, which is independent of $U_1 = \bar{Y}$, by part (b).

17. (5880*)

(a) If $U_1 = Y_1/\sqrt{Y_2/r}$ and $U_2 = Y_2$, then $Y_2 = U_2$ and $Y_1 = U_1\sqrt{U_2/r}$, so that

$$J = \begin{vmatrix} \frac{dy_1}{du_1} & \frac{dy_1}{du_2} \\ \frac{dy_2}{du_1} & \frac{dy_2}{du_2} \end{vmatrix} = \begin{vmatrix} \sqrt{u_2/r} & \frac{u_1}{2\sqrt{u_2 r}} \\ 0 & 1 \end{vmatrix} = \sqrt{u_2/r}$$

as desired.

(b) The joint pdf of (U_1, U_2) is obtained by multiplying a product of normal and χ^2 pdfs with J , as

$$\begin{aligned} f_{U_1, U_2}(u_1, u_2) &= f_{Y_1, Y_2}(y_1, y_2)|J| = f_{Y_1}(y_1)f_{Y_2}(y_2)|J| \\ &= \frac{1}{\sqrt{2\pi}} e^{-y_1^2/2} \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} y_2^{\frac{r}{2}-1} e^{-y_2/2} \sqrt{u_2/r} \\ &= \frac{1}{\sqrt{2\pi}} e^{-u_1^2 u_2 / r / 2} \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} u_2^{\frac{r}{2}-1} e^{-u_2/2} \sqrt{u_2/r} \\ &= \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}} \sqrt{2} \sqrt{\pi} \sqrt{r}} \sqrt{u_2} u_2^{\frac{r}{2}-1} e^{-u_1^2 u_2 / r / 2} e^{-u_2/2} \\ &= \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r+1}{2}} \sqrt{\pi r}} u_2^{\frac{r+1}{2}-1} e^{-\frac{[1+(u_1^2/r)]u_2}{2}} \end{aligned}$$

for $-\infty < u_1 < \infty$ and $u_2 > 0$.

(c) Using the hint,

$$\begin{aligned} f_{U_1}(u_1) &= \int f_{U_1, U_2}(u_1, u_2) du_2 = \int_0^\infty \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r+1}{2}} \sqrt{\pi r}} u_2^{\frac{r+1}{2}-1} e^{-\frac{[1+(u_1^2/r)]u_2}{2}} du_2 \\ &= \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r+1}{2}} \sqrt{\pi r}} \int_0^\infty u_2^{\frac{r+1}{2}-1} e^{-\frac{[1+(u_1^2/r)]u_2}{2}} du_2 \\ &= \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r+1}{2}} \sqrt{\pi r}} \Gamma\left(\frac{r+1}{2}\right) \left(\frac{2}{[1+(u_1^2/r)]}\right)^{\frac{r+1}{2}} \\ &\quad \times \underbrace{\int_0^\infty \frac{1}{\Gamma(\frac{r+1}{2})} \left(\frac{[1+(u_1^2/r)]}{2}\right)^{\frac{r+1}{2}} u_2^{\frac{r+1}{2}-1} e^{-\frac{[1+(u_1^2/r)]u_2}{2}} du_2}_{=1 \text{ (integrates to 1 by noting the gamma pdf)}} \\ &= \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})\sqrt{\pi r}} \frac{1}{[1+(u_1^2/r)]^{\frac{r+1}{2}}}, \quad -\infty < u_1 < \infty \end{aligned}$$

This is the pdf of the t -distribution with r d.f.

18. (5880*)

(a) Suppose that

$$U_1 = \frac{Y_1}{Y_2}$$

where $Y_1 \sim \chi^2(m)$ and $Y_2 \sim \chi^2(n)$ are independent, and $Y_1 > 0$ and $Y_2 > 0$. If we let $U_2 = Y_2$, then $Y_2 = U_2$ and $Y_1 = U_1 U_2$, so that

$$J = \left| \begin{array}{cc} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{array} \right| = \left| \begin{array}{cc} u_2 & u_1 \\ 0 & 1 \end{array} \right| = u_2.$$

(b) Since

$$f_{Y_1}(y_1) = \frac{1}{\Gamma(m/2)2^{m/2}} y_1^{m/2-1} e^{-y_1/2} \quad \text{and} \quad f_{Y_2}(y_2) = \frac{1}{\Gamma(n/2)2^{n/2}} y_2^{n/2-1} e^{-y_2/2}$$

it follows that

$$\begin{aligned} f_{U_1, U_2}(u_1, u_2) &= f_{Y_1, Y_2}(y_1, y_2) |J| = f_{Y_1}(y_1) f_{Y_2}(y_2) |J| \\ &= \frac{1}{\Gamma(m/2)2^{m/2}} y_1^{m/2-1} e^{-y_1/2} \frac{1}{\Gamma(n/2)2^{n/2}} y_2^{n/2-1} e^{-y_2/2} |J| \\ &= \frac{1}{\Gamma(m/2)2^{m/2}} (u_1 u_2)^{m/2-1} e^{-(u_1 u_2)/2} \frac{1}{\Gamma(n/2)2^{n/2}} u_2^{n/2-1} e^{-u_2/2} \cdot u_2 \\ &= \frac{1}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} u_1^{m/2-1} u_2^{(m+n)/2-1} e^{-(1+u_1)u_2/2} \end{aligned}$$

with $u_1 > 0$ and $u_2 > 0$.

(c) The marginal of U_1 is obtained by

$$\begin{aligned} f_{U_1}(u_1) &= \int f_{U_1, U_2}(u_1, u_2) du_2 \\ &= \int_0^\infty \frac{1}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} u_1^{m/2-1} u_2^{(m+n)/2-1} e^{-(1+u_1)u_2/2} du_2 \\ &= \frac{1}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} u_1^{m/2-1} \int_0^\infty u_2^{(m+n)/2-1} e^{-(1+u_1)u_2/2} du_2 \\ &= \frac{1}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} u_1^{m/2-1} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\left(\frac{1+u_1}{2}\right)^{(m+n)/2}} \\ &\quad \times \underbrace{\int_0^\infty \frac{\left(\frac{1+u_1}{2}\right)^{(m+n)/2}}{\Gamma\left(\frac{m+n}{2}\right)} u_2^{(m+n)/2-1} e^{-(1+u_1)u_2/2} du_2}_{=1} \\ &= \frac{1}{\Gamma(m/2)\Gamma(n/2)2^{(m+n)/2}} u_1^{m/2-1} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\left(\frac{1+u_1}{2}\right)^{(m+n)/2}} \\ &= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{u_1^{m/2-1}}{(1+u_1)^{(m+n)/2}} \end{aligned}$$

Finally, let

$$W = \frac{Y_1/m}{Y_2/n} = \frac{n}{m} \frac{Y_1}{Y_2} = \frac{n}{m} U_1$$

then $U_1 = \left(\frac{m}{n}\right) W$ so that $\frac{du_1}{dw} = \frac{m}{n}$. Hence,

$$\begin{aligned} f_W(w) &= f_{U_1}(u_1) \left| \frac{du_1}{dw} \right| \\ &= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \frac{u_1^{p/2-1}}{(1+u_1)^{(m+n)/2}} |J| \\ &= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \frac{\left(\left(\frac{m}{n}\right) w\right)^{m/2-1}}{\left(1 + \left(\frac{m}{n}\right) w\right)^{(m+n)/2}} \cdot \frac{m}{n} \\ &= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} \frac{w^{m/2-1}}{\left(1 + \left(\frac{m}{n}\right) w\right)^{(m+n)/2}} \end{aligned}$$

for $w > 0$. This is the pdf of $F(m, n)$, as was to be seen.

19. **(5880*)** First, note that the cdf of $X_{(n)}$ can be computed by

$$\begin{aligned} P(X_{(n)} \leq x) &= P(X_1 \leq x, \dots, X_n \leq x) = P(X_1 \leq x) \cdots P(X_n \leq x) = [P(X_1 \leq x)]^n \\ &= \left(\frac{x}{\theta}\right)^n, \quad 0 < x < \theta \end{aligned}$$

If $Y_n = n(\theta - X_{(n)})$,

$$\begin{aligned} P(Y_n \leq y) &= P(n(\theta - X_{(n)}) \leq y/n) = P(\theta - X_{(n)} \leq y/n) \\ &= P(X_{(n)} \geq \theta - y/n) = 1 - P(X_{(n)} < \theta - y/n) \\ &= 1 - \left(\frac{\theta - y/n}{\theta}\right)^n = 1 - \left(1 - \frac{y}{\theta n}\right)^n \rightarrow 1 - e^{-y/\theta} \end{aligned}$$

which we recognize as the cdf of $\text{Exponential}(\theta)$ (i.e., if $Y \sim \text{Exponential}(\theta)$, then we have that $P(Y_n \leq y) \rightarrow 1 - e^{-y/\theta} = F_Y(y) = P(Y \leq y)$ as $n \rightarrow \infty$). Hence Y_n converges in distribution to $Y \sim \text{Exponential}(\theta)$.

20. **(5880*)**

(a) Assuming that $\mu = 0$,

$$U_n = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^n Y_i - n\mu}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^n Y_i}{\sqrt{n}\sigma}$$

Then, since $Y_1, \dots, Y_n$ are iid,

$$\begin{aligned} M_{U_n}(t) &= E[e^{tU_n}] = E[\exp(tU_n)] = E\left[\exp\left(t \frac{\sum_{i=1}^n Y_i}{\sqrt{n}\sigma}\right)\right] = E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} \sum_{i=1}^n Y_i\right)\right] \\ &= E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} Y_1 + \cdots + \frac{t}{\sqrt{n}\sigma} Y_n\right)\right] = E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} Y_1\right)\right] \cdots E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} Y_n\right)\right] \\ &= \left(E\left[\exp\left(\frac{t}{\sqrt{n}\sigma} Y_1\right)\right]\right)^n = \left(M_{Y_1}\left(\frac{t}{\sqrt{n}\sigma}\right)\right)^n \end{aligned}$$

(b) If each Y_i has pdf $f_Y(y)$, we have that

$$\begin{aligned}
M_{Y_1} \left(\frac{t}{\sqrt{n}\sigma} \right) &= E \left[\exp \left(\frac{t}{\sqrt{n}\sigma} Y_1 \right) \right] = \int \exp \left(\frac{t}{\sqrt{n}\sigma} y \right) f_Y(y) dy \\
&= \int \left[1 + \frac{ty}{\sqrt{n}\sigma} + \frac{t^2 y^2}{2!n\sigma^2} + \cdots \right] f_Y(y) dy \\
&\quad \text{(using the fact that } e^a = 1 + a + a^2/2! + \cdots \text{)} \\
&= \int f_Y(y) dy + \int \frac{ty}{\sqrt{n}\sigma} f_Y(y) dy + \int \frac{t^2 y^2}{2!n\sigma^2} f_Y(y) dy + \cdots \\
&= \underbrace{\int f_Y(y) dy}_{=1} + \frac{t}{\sqrt{n}\sigma} \underbrace{\int y f_Y(y) dy}_{=E(Y_i)=\mu=0} + \frac{t^2}{2!n\sigma^2} \underbrace{\int y^2 f_Y(y) dy}_{=E(Y_i^2)=\sigma^2} + \cdots \\
&\quad \text{(why?)} \\
&= 1 + 0 + \frac{t^2}{2!n\sigma^2} \sigma^2 + \cdots \\
&= 1 + \frac{t^2}{2n} + \cdots
\end{aligned}$$

(c) Assuming that the “ $\cdots$ ” terms are small, we can reduce the part (b) to

$$M_{Y_1} \left(\frac{t}{\sqrt{n}\sigma} \right) = 1 + \frac{t^2}{2n}$$

Thus,

$$M_{U_n}(t) = \left(E \left[\exp \left(\frac{t}{\sqrt{n}\sigma} Y_1 \right) \right] \right)^n = \left(M_{Y_1} \left(\frac{t}{\sqrt{n}\sigma} \right) \right)^n = \left(1 + \frac{t^2}{2n} \right)^n \rightarrow e^{t^2/2}$$

by a result in calculus ($(1 + a/n)^n \rightarrow e^a$ as $n \rightarrow \infty$). Now, $e^{t^2/2}$ is the mgf of $N(0, 1)$, and if $Z \sim N(0, 1)$, then we can let $M_Z(t) = e^{t^2/2}$, so that we can conclude that

$$M_{U_n}(t) \rightarrow M_Z(t)$$

or equivalently,

$$U_n \rightarrow Z$$

in distribution.