Homework 5 Solution

1. Using the hint,

$$\begin{aligned}
\text{MSE}(\hat{\theta}) &= E[(\hat{\theta}-\theta)^2] = E[(\hat{\theta}-E(\hat{\theta})+E(\hat{\theta})-\theta)^2] \\
&= E[(\hat{\theta}-E(\hat{\theta}))^2 + (E(\hat{\theta})-\theta)^2 + 2(\hat{\theta}-E(\hat{\theta}))(E(\hat{\theta})-\theta)] \\
&= E[(\hat{\theta}-E(\hat{\theta}))^2] + E[(E(\hat{\theta})-\theta)^2] + E[2(\hat{\theta}-E(\hat{\theta}))(E(\hat{\theta})-\theta)] \\
&= \text{Var}(\hat{\theta}) + (E(\hat{\theta})-\theta)^2 + 2\underbrace{[E(\hat{\theta})-E(\hat{\theta})]}_{=0}(E(\hat{\theta})-\theta) \\
&= \text{Var}(\hat{\theta}) + [\text{Bias}(\hat{\theta})]^2
\end{aligned}$$

2. (a) Since $\hat{\theta}_4 = \min\{Y_1, Y_2, Y_3\} = Y_{(1)}$, and the cdf of Y is $F_Y(y) = 1 - e^{-y/\theta}$, $y > 0$, we have for the kth order statistic

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!}[F_Y(y)]^{k-1}[1-F_Y(y)]^{n-k} f_Y(y)$$

and so

$$f_{Y_{(1)}}(y) = \frac{3!}{0!2!}[1-e^{-y/\theta}]^0[1-(1-e^{-y/\theta})]^2\frac{1}{\theta}e^{-y/\theta} = \frac{3}{\theta}e^{-3y/\theta}$$

for $y > 0$ (note that it is the pdf of Exponential($\theta/3$)). Thus, $E(\hat{\theta}_4) = E(Y_{(1)}) = \theta/3 \neq \theta$.

(b) The unbiased estimators are

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1+Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1+2Y_2}{3}, \quad \hat{\theta}_5 = \overline{Y}$$

Since Y_1, Y_2, Y_3 are iid, and $\text{Var}(Y_1) = \theta^2$, we have that

$$\begin{aligned}
\text{Var}(\hat{\theta}_1) &= \text{Var}(Y_1) = \theta^2 \\
\text{Var}(\hat{\theta}_2) &= \text{Var}\left(\frac{Y_1+Y_2}{2}\right) = \frac{1}{2^2}\text{Var}(Y_1) + \frac{1}{2^2}\text{Var}(Y_2) = \frac{\theta^2}{2} \\
\text{Var}(\hat{\theta}_3) &= \text{Var}\left(\frac{Y_1+2Y_2}{3}\right) = \frac{1}{3^2}\text{Var}(Y_1) + \frac{2^2}{3^2}\text{Var}(Y_2) = \frac{5\theta^2}{9} \\
\text{Var}(\hat{\theta}_5) &= \text{Var}(\overline{Y}) = \frac{\text{Var}(Y_1)}{3} = \frac{\theta^2}{3}
\end{aligned}$$

Therefore, $\hat{\theta}_5$ has the smallest variance.

3. (a) We have

$$E(\hat{\theta}_3) = E(a\hat{\theta}_1 + (1-a)\hat{\theta}_2) = aE(\hat{\theta}_1) + (1-a)E(\hat{\theta}_2) = a\theta + (1-a)\theta = \theta$$

(b) If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, then

$$\text{Var}(\hat{\theta}_3) = \text{Var}(a\hat{\theta}_1 + (1-a)\hat{\theta}_2) = \text{Var}(a\hat{\theta}_1) + \text{Var}((1-a)\hat{\theta}_2) = a^2\text{Var}(\hat{\theta}_1) + (1-a)^2\text{Var}(\hat{\theta}_2)$$

(c) To find the value of a that minimizes $\text{Var}(\hat{\theta}_3)$, we take

$$\begin{aligned}\frac{d}{da}\text{Var}(\hat{\theta}_3) &= \frac{d}{da}[a^2\text{Var}(\hat{\theta}_1) + (1-a)^2\text{Var}(\hat{\theta}_2)] = \frac{d}{da}[a^2\sigma_1^2 + (1-a)^2\sigma_2^2] \\ &= 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 2a(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0\end{aligned}$$

so that

$$a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

To check that it is indeed a minimizing value, we see that

$$\frac{d^2}{da^2}\text{Var}(\hat{\theta}_3) = \frac{d}{da}[2a(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2] = 2(\sigma_1^2+\sigma_2^2) > 0$$

since $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$.

4. (a) The cdf of Y is given by

$$F_Y(y) = \int_0^y \frac{\alpha t^{\alpha-1}}{\theta^\alpha}\,dt = \left(\frac{t}{\theta}\right)^\alpha\Bigg|_0^y = \left(\frac{y}{\theta}\right)^\alpha, \quad 0 \le y \le \theta$$

(b) Since $\hat{\theta} = \max\{Y_1, Y_2, \ldots, Y_n\} = Y_{(n)}$, we have that

$$f_{Y_{(n)}}(y) = n[F_Y(y)]^{n-1} f_Y(y) = n\left(\frac{y}{\theta}\right)^{\alpha(n-1)} \frac{\alpha y^{\alpha-1}}{\theta^\alpha} = \alpha n \frac{y^{\alpha n-1}}{\theta^{\alpha n}}$$

for $0 \le y \le \theta$. Thus,

$$\begin{aligned}E(\hat{\theta}) &= E(Y_{(n)}) = \int y f_{Y_{(n)}}(y)\,dy = \int_0^\theta y\alpha n \frac{y^{\alpha n-1}}{\theta^{\alpha n}}\,dy = \int_0^\theta \alpha n \frac{y^{\alpha n}}{\theta^{\alpha n}}\,dy \\ &= \left(\frac{\alpha n}{\alpha n+1}\right)\left(\frac{y^{\alpha n+1}}{\theta^{\alpha n}}\right)\Bigg|_0^\theta = \left(\frac{\alpha n}{\alpha n+1}\right)\theta\end{aligned}$$

and hence $\hat{\theta}$ is a biased estimator for θ.

(c) It is easy see that

$$\left(\frac{\alpha n+1}{\alpha n}\right)\hat{\theta}$$

is an unbiased estimator of θ.

5. Consider $\hat{\theta} = Y_{(1)} = \min\{Y_1, Y_2, \ldots, Y_n\}$. Since

$$f_Y(y) = \frac{1}{\theta} \quad \text{and} \quad F_Y(y) = \int_0^y \frac{1}{\theta}\,dt = \frac{y}{\theta}, \quad 0 < y < \theta$$

we see that

$$f_{Y_{(1)}}(y) = n[1-F_Y(y)]^{n-1} f_Y(y) = n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{n}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1}, \quad 0 < y < \theta$$

and using integration by parts, we obtain

$$E(\hat{\theta}) = E(Y_{(1)}) = \int y f_{Y_{(1)}}(y)\, dy = \int_0^\theta y \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1}\, dy = \frac{\theta}{n+1}$$

Now, again using integration by parts (twice),

$$\begin{aligned}
\mathrm{Var}(\hat{\theta}) &= \mathrm{Var}(Y_{(1)}) = E(Y_{(1)}^2) - [E(Y_{(1)})]^2 = \int y^2 f_{Y_{(1)}}(y)\, dy - \left[\frac{\theta}{n+1}\right]^2 \\
&= \int_0^\theta y^2 \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1}\, dy - \frac{\theta^2}{(n+1)^2} \\
&= \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} \\
&= \frac{n\theta^2}{(n+1)^2(n+2)}
\end{aligned}$$

From $E(\hat{\theta}) = E(Y_{(1)}) = \theta/(n+1)$ above, we conclude that an unbiased estimator for θ is given by $(n+1)\hat{\theta} = (n+1)Y_{(1)}$.

6. (a) Similar to previous problem, we have that

$$f_{Y_{(1)}}(y) = n[1 - F_Y(y)]^{n-1} f_Y(y) = n[1 - (1 - e^{-y/\theta})]^{n-1} \frac{1}{\theta} e^{-y/\theta} = \frac{n}{\theta} e^{-ny/\theta} = \frac{n}{\theta} e^{-(n/\theta)y}$$

for $y > 0$, so that $Y_{(1)} \sim$ Exponential(θ/n). Thus, $E(Y_{(1)}) = \theta/n$, and if $\hat{\theta}_1 = nY_{(1)}$, then

$$E(\hat{\theta}_1) = E(nY_{(1)}) = n\frac{\theta}{n} = \theta$$

Now, again using that $Y_{(1)} \sim$ Exponential(θ/n),

$$\mathrm{Var}(\hat{\theta}_1) = \mathrm{Var}(nY_{(1)}) = n^2 \mathrm{Var}(Y_{(1)}) = n^2 \frac{\theta^2}{n^2} = \theta^2$$

and since $\hat{\theta}_1$ is unbiased for θ, we conclude that

$$\mathrm{MSE}(\hat{\theta}_1) = \mathrm{Var}(\hat{\theta}_1) = \theta^2$$

(b) Since we recognize that $Y_1, Y_2, \ldots, Y_n$ are iid Exponential(θ), we have that

$$E(\hat{\theta}_2) = E(\overline{Y}) = E(Y_i) = \theta$$

so that $\hat{\theta}_2$ is unbiased for θ as well.

(c) We have also seen that $\mathrm{Var}(\hat{\theta}_1) = \mathrm{Var}(nY_{(1)}) = \theta^2$. For $\hat{\theta}_2 = \overline{Y}$, we have that

$$\mathrm{Var}(\hat{\theta}_2) = \mathrm{Var}(\overline{Y}) = \frac{\theta^2}{n}$$

Consequently,

$$\mathrm{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)} = \frac{\theta^2/n}{\theta^2} = \frac{1}{n}$$

7. (a) If $X \sim \chi^2(v)$, then $E(X) = v$ and $\text{Var}(X) = 2v$. Now, since $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$ and $Y_1 - Y_2 \sim N(0, 2\sigma^2) \implies (Y_1 - Y_2)^2/(2\sigma^2) \sim \chi^2(1)$, it follows that

$$\text{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1) \implies \text{Var}(\hat{\sigma}_1^2) = \text{Var}(S^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{(n-1)}$$

and

$$\text{Var}\left(\frac{(Y_1 - Y_2)^2}{2\sigma^2}\right) = 2 \implies \text{Var}(\hat{\sigma}_2^2) = \text{Var}\left(\frac{(Y_1 - Y_2)^2}{2}\right) = 2\sigma^4$$

(b) The efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is given by

$$\text{eff}(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \frac{\text{Var}(\hat{\sigma}_2^2)}{\text{Var}(\hat{\sigma}_1^2)} = \frac{2\sigma^4}{2\sigma^4/(n-1)} = (n-1)$$

8. (a) Since $E(Y) = np$ and

$$E(\hat{p}_2) = E\left(\frac{Y+1}{n+2}\right) = \frac{E(Y)+1}{n+2} = \frac{np+1}{n+2}$$

the bias of $\hat{p}_2$ is given by

$$\text{Bias}(\hat{p}_2) = E(\hat{p}_2) - p = \frac{np+1}{n+2} - p = \frac{1-2p}{n+2}$$

(b) Note that $\text{Var}(Y) = np(1-p)$. Since $\hat{p}_1 = Y/n$ is unbiased,

$$\text{MSE}(\hat{p}_1) = \text{Var}(\hat{p}_1) = \text{Var}\left(\frac{Y}{n}\right) = \frac{1}{n^2} np(1-p) = \frac{p(1-p)}{n}$$

For $\hat{p}_2 = (Y+1)/(n+2)$, we first see that

$$\text{Var}(\hat{p}_2) = \text{Var}\left(\frac{Y+1}{n+2}\right) = \frac{1}{(n+2)^2}\text{Var}(Y) = \frac{np(1-p)}{(n+2)^2}$$

and therefore

$$\text{MSE}(\hat{p}_2) = \text{Var}(\hat{p}_2) + [\text{Bias}(\hat{p}_2)]^2 = \frac{np(1-p)}{(n+2)^2} + \left(\frac{1-2p}{n+2}\right)^2 = \frac{np(1-p) + (1-2p)^2}{(n+2)^2}$$

9. (a) First,

$$\text{Var}(\hat{\theta}_1) = \text{Var}((n+1)Y_{(1)}) = (n+1)^2\text{Var}(Y_{(1)}) = (n+1)^2 \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{n\theta^2}{(n+2)}$$

For $\text{Var}(\hat{\theta}_2)$, recall Example 9.1 that

$$\text{Var}(\hat{\theta}_2) = \text{Var}\left(\left(\frac{n+1}{n}\right) Y_{(n)}\right) = \frac{\theta^2}{n(n+2)}$$

and therefore,

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{\text{Var}(\hat{\theta}_2)}{\text{Var}(\hat{\theta}_1)} = \frac{\theta^2/[n(n+2)]}{n\theta^2/(n+2)} = \frac{1}{n^2}$$

(b) Since $\hat{\theta}_2$ is unbiased and

$$\text{Var}(\hat{\theta}_2) = \text{Var}\left(\left(\frac{n+1}{n}\right) Y_{(n)}\right) = \frac{\theta^2}{n(n+2)}$$

(see Example 9.1), we see that

$$\lim_{n\to\infty} \text{Var}(\hat{\theta}_2) = \lim_{n\to\infty} \frac{\theta^2}{n(n+2)} = 0$$

Thus, it follows from Theorem 9.1 that $\hat{\theta}_2$ is a consistent estimator for θ. Now, since

$$Y_{(n)} = \frac{n}{(n+1)}\hat{\theta}_2$$

and $n/(n+1) \to 1$ and $\hat{\theta}_2 \xrightarrow{p} \theta$ as $n \to \infty$, we have that

$$Y_{(n)} = \frac{n}{(n+1)}\hat{\theta}_2 \xrightarrow{p} 1 \cdot \theta = \theta$$

by Theorem 9.2, and so that $Y_{(n)}$ is a consistent estimator for θ. To show that $Y_{(n)}$ is asymptotically unbiased estimator for θ, note that

$$\text{Bias}(Y_{(n)}) = E(Y_{(n)}) - \theta = \left(\frac{n}{n+1}\right)\theta - \theta \to 0$$

as $n \to \infty$, proving the result. (Note that the last two results can be solved in many different ways).

10. If $Y \sim \text{Binomial}(n, p)$, then $E(Y) = np$ and $\text{Var}(Y) = np(1-p)$, so that Y/n is unbiased for p, and since $\text{Var}(Y/n) = p(1-p)/n \to 0$ as $n \to \infty$, then Y/n is consistent for p as well. Now, write

$$\hat{p} = \frac{Y+1}{n+2} = \frac{Y}{n+2} + \frac{1}{n+2} = \frac{n}{n+2}\frac{Y}{n} + \frac{1}{n+2}$$

Since $Y/n \xrightarrow{p} p$, with $n/(n+2) \to 1$ and $1/(n+2) \to 0$ as $n \to \infty$, it follows that

$$\hat{p} = \frac{Y+1}{n+2} = \frac{n}{n+2}\frac{Y}{n} + \frac{1}{n+2} \xrightarrow{p} 1 \cdot p + 0 = p$$

by Theorem 9.2. Hence, $\hat{p}$ a consistent estimator for p. From

$$E(\hat{p}) = E\left(\frac{Y+1}{n+2}\right) = \frac{n}{n+2}E\left(\frac{Y}{n}\right) + \frac{1}{n+2} = \frac{n}{n+2}p + \frac{1}{n+2}$$

it is easy to see that $E(\hat{p}) \to p$ as $n \to \infty$, so that $\hat{p}$ an asymptotically unbiased estimator for p as well. (Again, this problem can be solved in many different ways)

11. By weak law of large numbers (WLLN), the conditions given in the problem ensure that

$$\overline{X} \xrightarrow{p} \mu_1 \quad \text{and} \quad \overline{Y} \xrightarrow{p} \mu_2$$

as $n \to \infty$, and by Theorem 9.2,

$$\overline{X} - \overline{Y} \xrightarrow{p} \mu_1 - \mu_2$$

as $n \to \infty$. Hence, $\overline{X} - \overline{Y}$ is a consistent estimator of $\mu_1 - \mu_2$.

12. If $Y_1, Y_2, \ldots, Y_n$ are iid Gamma(α, β), then $E(Y_i) = \alpha\beta$ and $\text{Var}(Y_i) = \alpha\beta^2 < \infty$. Now, since $E(\overline{Y}) = E(Y_i) = \alpha\beta$ and $\text{Var}(\overline{Y}) = \alpha\beta^2/n \to 0$ as $n \to \infty$, it follows by Theorem 9.1 that $\overline{Y}$ is a consistent estimator of $\alpha\beta$.

13. **(5880*)** If $X \sim \chi^2(v)$, then $E(X) = v$ and $\text{Var}(X) = 2v$. Now, since $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$, it follows that

$$\text{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

which implies

$$\text{Var}(S^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{(n-1)}$$

so that

$$\text{MSE}(S^2) = \text{Var}(S^2) + [\text{Bias}(S^2)]^2 = \text{Var}(S^2) + 0 = \frac{2\sigma^4}{(n-1)}$$

Now, note that

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \frac{1}{n}\frac{(n-1)}{(n-1)}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = \frac{(n-1)}{n}S^2$$

so that

$$E(\hat{\sigma}_n^2) = E\left(\frac{(n-1)}{n}S^2\right) = \frac{(n-1)}{n}E(S^2) = \frac{(n-1)\sigma^2}{n}$$

and that

$$\text{Var}(\hat{\sigma}_n^2) = \text{Var}\left(\frac{(n-1)}{n}S^2\right) = \frac{(n-1)^2}{n^2}\text{Var}(S^2) = \frac{(n-1)^2}{n^2}\frac{2\sigma^4}{(n-1)} = \frac{2(n-1)\sigma^4}{n^2}$$

Hence,

$$\begin{aligned}\text{MSE}(\hat{\sigma}_n^2) &= \text{Var}(\hat{\sigma}_n^2) + [\text{Bias}(\hat{\sigma}_n^2)]^2 = \frac{2(n-1)\sigma^4}{n^2} + [E(\hat{\sigma}_n^2) - \sigma^2]^2 \\ &= \frac{2(n-1)\sigma^4}{n^2} + \left[\frac{(n-1)\sigma^2}{n} - \sigma^2\right]^2 = \frac{2(n-1)\sigma^4}{n^2} + \frac{\sigma^4}{n^2} \\ &= \frac{(2n-1)\sigma^4}{n^2}\end{aligned}$$

(You can check that $\text{MSE}(\hat{\sigma}_n^2) < \text{MSE}(S^2)$, which implies that being unbiased is not always better!).

14. **(5880*)**

(a) If $U = Y^2$, then $Y = \sqrt{U}$ (since $Y > 0$) and $\frac{dy}{du} = \frac{1}{2}u^{-1/2}$, so that

$$f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| = \left(\frac{2y}{\theta}\right)e^{-y^2/\theta}\left|\frac{1}{2}u^{-1/2}\right| = \left(\frac{2u^{1/2}}{\theta}\right)e^{-u/\theta}\frac{1}{2}u^{-1/2} = \frac{1}{\theta}e^{-u/\theta}$$

for $u > 0$ (note that $U \sim$ Exponential(θ)).

(b) Since $W_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 = \frac{1}{n}\sum_{i=1}^{n} U_i$, where $U_i \sim$ Exponential(θ) are iid with $E(U_i) = \theta$, it follows from WLLN that W_n is a consistent estimator for θ.