Homework 6 Solution

1. Recall that if $Y \sim$ Geometric(p), then $P(Y = y) = (1-p)^{y-1}p$, $y = 1, 2, \ldots$. Then, since

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} P(Y_i = y_i) = \prod_{i=1}^{n} [(1-p)^{y_i - 1}p] = (1-p)^{\sum_{i=1}^{n}(y_i-1)}p^n = (1-p)^{\sum_{i=1}^{n} y_i - n}p^n \\ &= (1-p)^{n\overline{y}-n}p^n = g\left(\overline{y}, p\right) h(y_1, \ldots, y_n) \end{aligned}$$

where

$$g\left(\overline{y}, p\right) = (1-p)^{n\overline{y}-n}p^n \quad \text{and} \quad h(y_1, \ldots, y_n) = 1$$

it follows that $\overline{Y}$ is a sufficient statistic for p.

2. Since

$$L(\theta) = \prod_{i=1}^{n} P(Y_i = y_i) = \prod_{i=1}^{n} \left(\frac{e^{-\theta}\theta^{y_i}}{y_i!} \right) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i}}{y_1! \cdots y_n!} = g\left(\sum_{i=1}^{n} y_i, \theta \right) h(y_1, \ldots, y_n)$$

where

$$g\left(\sum_{i=1}^{n} y_i, \theta \right) = e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i} \quad \text{and} \quad h(y_1, \ldots, y_n) = \frac{1}{y_1! \cdots y_n!}$$

it follows that $\sum_{i=1}^{n} Y_i$ is a sufficient statistic for θ.

Now, $E(\overline{Y}) = E(Y) = \theta$ and $\overline{Y} = \sum_{i=1}^{n} Y_i/n$ is a function of a sufficient statistic $\sum_{i=1}^{n} Y_i$, and $Y \sim$ Poisson(θ) belongs to an exponential family*, so we conclude that $\overline{Y}$ is the UMVUE of θ. (Footnote: *See last page; also applies to Problems 3 and 4).

3. First,

$$\begin{aligned} L(\beta) &= \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} \left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}} y_i^{\alpha-1} e^{-y_i/\beta} \right) = \frac{\prod_{i=1}^{n} y_i^{\alpha-1} e^{-\sum_{i=1}^{n} y_i/\beta}}{[\Gamma(\alpha)]^n \beta^{n\alpha}} \\ &= \frac{e^{-\sum_{i=1}^{n} y_i/\beta}}{[\Gamma(\alpha)]^n \beta^{n\alpha}} \prod_{i=1}^{n} y_i^{\alpha-1} = g\left(\sum_{i=1}^{n} y_i, \beta \right) h(y_1, \ldots, y_n) \end{aligned}$$

where

$$g\left(\sum_{i=1}^{n} y_i, \beta \right) = \frac{e^{-\sum_{i=1}^{n} y_i/\beta}}{[\Gamma(\alpha)]^n \beta^{n\alpha}} \quad \text{and} \quad h(y_1, \ldots, y_n) = \prod_{i=1}^{n} y_i^{\alpha-1}$$

it follows that $\sum_{i=1}^{n} Y_i$ is a sufficient statistic for β. Now, since $E(\overline{Y}) = E(Y) = \alpha\beta$, then $\overline{Y}/\alpha$ is an unbiased estimator for β (recall that α is known). And because $\overline{Y}/\alpha$ is a function of a sufficient statistic $\sum_{i=1}^{n} Y_i$, and $Y \sim$ Gamma(α, β) belongs to an exponential family*, we conclude that $\overline{Y}/\alpha$ is the UMVUE of β.

4. We know that $\overline{Y}$ is a sufficient statistic for θ. As in Example 9.9, we first attempt $\overline{Y}^2$, and obtain

$$E(\overline{Y}^2) = \text{Var}(\overline{Y}) + [E(\overline{Y})]^2 = \frac{1}{n} + \theta^2$$

which implied that

$$E\left(\overline{Y}^2 - \frac{1}{n}\right) = \theta^2$$

Since $\overline{Y}^2 - 1/n$ is a function of a sufficient statistic $\overline{Y}$, and $Y \sim N(\mu, \sigma^2)$ belongs to an exponential family*, we conclude that $\overline{Y}^2 - 1/n$ is the UMVUE of θ^2.

5. Here, we need to watch the region of Y, so that

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n f_{Y_i}(y_i) = \prod_{i=1}^n e^{-(y_i-\theta)} 1_{(y_i>\theta)} = e^{-\sum_{i=1}^n y_i + n\theta} \prod_{i=1}^n 1_{(y_i>\theta)} \\ &= e^{n\theta} 1_{(\theta<y_1,\ldots,y_n<\infty)} e^{-\sum_{i=1}^n y_i} = e^{n\theta} 1_{(\theta<y_{(1)}<\cdots<y_{(n)}<\infty)} e^{-\sum_{i=1}^n y_i} \\ &= e^{n\theta} 1_{(\theta<y_{(1)})} e^{-\sum_{i=1}^n y_i} = g(y_{(1)}, \theta) h(y_1, \ldots, y_n) \end{aligned}$$

where

$$g(y_{(1)}, \theta) = e^{n\theta} 1_{(\theta<y_{(1)})} \quad \text{and} \quad h(y_1, \ldots, y_n) = e^{-\sum_{i=1}^n y_i}$$

Hence, $Y_{(1)} = \min\{Y_1, \ldots, Y_n\}$ is a sufficient statistic for θ.

6. We have that

$$\begin{aligned} L(\theta) &= \prod_{i=1}^n f_{Y_i}(y_i) = \prod_{i=1}^n \frac{\alpha y_i^{\alpha-1}}{\theta^\alpha} 1_{(0\le y_i\le\theta)} = \frac{\alpha^n \prod_{i=1}^n y_i^{\alpha-1}}{\theta^{n\alpha}} 1_{(0\le y_1,\ldots,y_n\le\theta)} \\ &= \frac{\alpha^n \prod_{i=1}^n y_i^{\alpha-1}}{\theta^{n\alpha}} 1_{(0\le y_{(1)}<\cdots<y_{(n)}\le\theta)} = \frac{\alpha^n \prod_{i=1}^n y_i^{\alpha-1}}{\theta^{n\alpha}} 1_{(y_{(n)}\le\theta)} \\ &= \frac{1_{(y_{(n)}\le\theta)}}{\theta^{n\alpha}} \alpha^n \prod_{i=1}^n y_i^{\alpha-1} = g(y_{(n)}, \theta) h(y_1, \ldots, y_n) \end{aligned}$$

where

$$g(y_{(n)}, \theta) = \frac{1_{(y_{(n)}\le\theta)}}{\theta^{n\alpha}} \quad \text{and} \quad h(y_1, \ldots, y_n) = \alpha^n \prod_{i=1}^n y_i^{\alpha-1}$$

Hence, $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$ is a sufficient statistic for θ.

(Note again that α is a known quantity here, so it is treated as a constant).

7. We have that

$$\begin{aligned} L(\alpha, \beta) &= \prod_{i=1}^n f_Y(y_i) = \prod_{i=1}^n \left[\frac{1}{\Gamma(\alpha)\beta^\alpha} y_i^{\alpha-1} e^{-y_i/\beta}\right] \\ &= \frac{1}{[\Gamma(\alpha)]^n \beta^{n\alpha}} \prod_{i=1}^n y_i^{\alpha-1} e^{-[\sum_{i=1}^n y_i]/\beta} \\ &= g\left(\left(\prod_{i=1}^n y_i, \sum_{i=1}^n y_i\right), (\alpha, \beta)\right) h(y_1, \ldots, y_n) \end{aligned}$$

where $h(y_1, \ldots, y_n) = 1$ and

$$g\left(\left(\prod_{i=1}^{n} y_i, \sum_{i=1}^{n} y_i\right), (\alpha, \beta)\right) = \frac{1}{[\Gamma(\alpha)]^n \beta^{n\alpha}} \prod_{i=1}^{n} y_i^{\alpha-1} e^{-[\sum_{i=1}^{n} y_i]/\beta}$$

we conclude that

$$\left(\prod_{i=1}^{n} Y_i, \sum_{i=1}^{n} Y_i\right)$$

are sufficient statistics for (α, β).

8. Since $E(Y) = \theta$, we set

$$E(Y) = \theta = \overline{Y}$$

and hence $\hat{\theta} = \overline{Y}$ is the method of moments estimator for θ.

9. Since $E(Y) = \mu$ and $\text{Var}(Y) = \sigma^2$, we have $E(Y^2) = \sigma^2 + \mu^2$. So from a set of equations

$$\begin{aligned} E(Y) &= \bar{Y} \\ E(Y^2) &= \frac{1}{n}\sum_{i=1}^{n} Y_i^2 \end{aligned}$$

we get

$$E(Y) = \mu = \bar{Y} \quad \text{and} \quad E(Y^2) = \sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$$

and thus

$$\hat{\mu} = \bar{Y} \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - (\bar{Y})^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

(it is a good practice to verify the last equality, which gives us a useful form).

10. We set

$$E(Y) = \int_k^\infty y\theta k^\theta y^{-(\theta+1)}\, dy = \theta k^\theta \int_k^\infty y^{-\theta}\, dy = \frac{k\theta}{\theta - 1} = \bar{Y}$$

yielding the result

$$\hat{\theta} = \frac{\bar{Y}}{\bar{Y} - k}$$

11. (a) Since

$$L(\theta) = \prod_{i=1}^{n} P(Y_i = y_i) = \prod_{i=1}^{n}\left(\frac{e^{-\theta}\theta^{y_i}}{y_i!}\right) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i}}{y_1! \cdots y_n!} = g\left(\sum_{i=1}^{n} y_i, \theta\right) h(y_1, \ldots, y_n)$$

and

$$\ln L(\theta) = -n\theta + \sum_{i=1}^{n} y_i \ln\theta - \ln(y_1! \cdots y_n!)$$

setting

$$\frac{d}{d\theta}\ln L(\theta) = -n + \frac{\sum_{i=1}^{n} y_i}{\theta} = 0$$

we see that $\theta = \sum_{i=1}^n y_i/n$, or

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^n Y_i}{n} = \overline{Y}$$

(b) Since $P(Y=0) = e^{-\theta}$, it follows that the MLE of $P(Y=0)$ is given by

$$e^{-\hat{\theta}_{MLE}} = e^{-\overline{Y}}$$

12. (a) First,

$$L(\beta) = \prod_{i=1}^n f_{Y_i}(y_i) = \prod_{i=1}^n \left(\frac{1}{\Gamma(\alpha)\beta^\alpha} y_i^{\alpha-1} e^{-y_i/\beta} \right) = \frac{\prod_{i=1}^n y_i^{\alpha-1} e^{-\sum_{i=1}^n y_i/\beta}}{[\Gamma(\alpha)]^n \beta^{n\alpha}}$$

and

$$\ln L(\beta) = \ln \left(\prod_{i=1}^n y_i^{\alpha-1} \right) - \sum_{i=1}^n y_i/\beta - n \ln \Gamma(\alpha) - n\alpha \ln \beta$$

so by setting

$$\frac{d}{d\beta} \ln L(\theta) = \frac{\sum_{i=1}^n y_i}{\beta^2} - \frac{n\alpha}{\beta} = 0$$

we see that $\beta = \sum_{i=1}^n y_i/(n\alpha)$, so that

$$\hat{\beta}_{MLE} = \frac{\sum_{i=1}^n Y_i}{n\alpha}$$

(since α is known)

(b) Note that $E(Y) = \alpha\beta$, and since α is known, it follows that the MLE of $E(Y)$ is given by

$$\alpha\hat{\beta}_{MLE} = \alpha \frac{\sum_{i=1}^n Y_i}{n\alpha} = \frac{\sum_{i=1}^n Y_i}{n} = \overline{Y}$$

13. The likelihood function is computed by

$$L(\theta) = \prod_{i=1}^n f_{Y_i}(y_i) = \prod_{i=1}^n [\theta^2 y_i e^{-\theta y_i}] = \theta^{2n} \left(\prod_{i=1}^n y_i \right) e^{-\theta \sum_{i=1}^n y_i} = \theta^{2n} e^{-\theta \sum_{i=1}^n y_i} \left(\prod_{i=1}^n y_i \right)$$

To find the MLE of θ, we work with

$$\ln L(\theta) = \ln \left(\theta^{2n} \left(\prod_{i=1}^n y_i \right) e^{-\theta \sum_{i=1}^n y_i} \right) = 2n \ln \theta + \sum_{i=1}^n \ln y_i - \theta \sum_{i=1}^n y_i$$

and set

$$\frac{d}{d\theta} \ln L(\theta) = \frac{2n}{\theta} + 0 - \sum_{i=1}^n y_i = 0$$

and solving for θ, we obtain

$$\hat{\theta} = \frac{2n}{\sum_{i=1}^n Y_i}$$

as the MLE of θ.

14. (a) First

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} \left(\frac{r y_i^{r-1}}{\theta} e^{-y_i^r/\theta} \right) = \frac{\prod_{i=1}^{n} (r y_i^{r-1})}{\theta^n} e^{-\sum_{i=1}^{n} y_i^r/\theta} \\ &= \frac{e^{-\sum_{i=1}^{n} y_i^r/\theta}}{\theta^n} \prod_{i=1}^{n} (r y_i^{r-1}) = g\left(\sum_{i=1}^{n} y_i^r, \theta \right) h(y_1, \ldots, y_n) \end{aligned}$$

where

$$g\left(\sum_{i=1}^{n} y_i^r, \theta \right) = \frac{e^{-\sum_{i=1}^{n} y_i^r/\theta}}{\theta^n} \quad \text{and} \quad h(y_1, \ldots, y_n) = \prod_{i=1}^{n} (r y_i^{r-1})$$

it follows that $\sum_{i=1}^{n} Y_i^r$ is a sufficient statistic for θ (recall that r is known).

(b) By part (a)

$$L(\theta) = \frac{\prod_{i=1}^{n} (r y_i^{r-1})}{\theta^n} e^{-\sum_{i=1}^{n} y_i^r/\theta}$$

and

$$\ln L(\theta) = \ln \left(\prod_{i=1}^{n} (r y_i^{r-1}) \right) - n \ln \theta - \sum_{i=1}^{n} y_i^r/\theta$$

so, by setting

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} y_i^r}{\theta^2} = 0$$

it follows that $\theta = \sum_{i=1}^{n} y_i^r/n$, or

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} Y_i^r}{n}$$

15. (a) We have that

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} [\theta y_i^{-2} I_{(\theta \le y_i < \infty)}] = \theta^n I_{(\theta \le y_1, \ldots, y_n < \infty)} \prod_{i=1}^{n} y_i^{-2} \\ &= \theta^n I_{(\theta \le y_{(1)} \le \cdots \le y_{(n)} < \infty)} \prod_{i=1}^{n} y_i^{-2} = \theta^n I_{(\theta \le y_{(1)})} \prod_{i=1}^{n} y_i^{-2} \\ &= g(y_{(1)}, \theta) h(y_1, \ldots, y_n) \end{aligned}$$

Then by Factorization Theorem, with

$$g(y_{(1)}, \theta) = \theta^n I_{(\theta \le y_{(1)})} \quad \text{and} \quad h(y_1, \ldots, y_n) = \prod_{i=1}^{n} y_i^{-2}$$

we see that $Y_{(1)}$ is a sufficient statistic for θ.

(b) Since

$$L(\theta) = \theta^n I_{(0 < \theta \le y_{(1)})} \prod_{i=1}^{n} y_i^{-2}$$

the likelihood is the largest if θ is large as possible. But since we have a restriction that $\theta \leq y_{(1)}$, it follows that the MLE of θ is given by

$$\hat{\theta}_{MLE} = Y_{(1)}$$

(the same as sufficient statistic)

(c) Since

$$E(Y) = \int_{\theta}^{\infty} y f_Y(y)\, dy = \int_{\theta}^{\infty} y\theta y^{-2}\, dy = \int_{\theta}^{\infty} \theta y^{-1}\, dy = \theta \ln y \Big|_{\theta}^{\infty} = \infty$$

the expectation does not exist, and hence the method of moments estimator does not exist.

16. Using the indicator functions, we have

$$L(\theta) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} \left[\frac{2\theta^2}{y_i^3} 1_{(\theta \leq y_i < \infty)} \right] = \frac{2^n \theta^{2n}}{\prod_{i=1}^{n} y_i^3} 1_{(\theta \leq y_{(1)} \leq \cdots \leq y_{(n)} < \infty)}$$

and the likelihood function is the largest if the θ is as large as possible, that is,

$$\hat{\theta}_{MLE} = Y_{(1)}$$

Then it follow that

$$\hat{\theta}^2 = Y_{(1)}^2$$

is the MLE of θ^2

17. Using the indicator functions, we have

$$L(\theta) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} \left[\frac{3y_i^2}{\theta^3} 1_{(0 \leq y_i \leq \theta)} \right] = \frac{\prod_{i=1}^{n} (3y_i^2)}{\theta^{3n}} 1_{(0 \leq y_{(1)} \leq \cdots \leq y_{(n)} \leq \theta)}$$

and the likelihood function is the largest if the θ is as small as possible, that is,

$$\hat{\theta}_{MLE} = Y_{(n)}$$

is the MLE of θ.

18. **(5880*)** Here, both α and θ are unknown, so we now have that (from Problem 6),

$$\begin{aligned} L(\alpha, \theta) &= \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} \frac{\alpha y_i^{\alpha-1}}{\theta^{\alpha}} 1_{(0 \leq y_i \leq \theta)} = \frac{\alpha^n \prod_{i=1}^{n} y_i^{\alpha-1}}{\theta^{n\alpha}} 1_{(y_{(n)} \leq \theta)} \\ &= \frac{\alpha^n}{\theta^{n\alpha}} \prod_{i=1}^{n} y_i^{\alpha-1} 1_{(y_{(n)} \leq \theta)} \\ &= g\left(\left(\prod_{i=1}^{n} y_i, y_{(n)} \right), (\alpha, \theta) \right) h(y_1, \ldots, y_n) \end{aligned}$$

where $h(y_1, \ldots, y_n) = 1$ and

$$g\left(\left(\prod_{i=1}^{n} y_i, y_{(n)}\right), (\alpha, \theta)\right) = \frac{\alpha^n}{\theta^{n\alpha}} \prod_{i=1}^{n} y_i^{\alpha-1} 1_{(y_{(n)} \leq \theta)}$$

Hence,

$$\left(\prod_{i=1}^{n} Y_i,\ Y_{(n)}\right)$$

are sufficient statistics for (α, θ).

19. **(5880*)**

(a) First, if $Y \sim$ Exponential(θ), then we have seen that $\hat{\theta} = \overline{Y}$ is the MLE of θ, so that $t(\hat{\theta}) = \hat{\theta} = \overline{Y}$ and $t(\theta) = \theta$. Hence,

$$\left[\frac{\partial t(\theta)}{\partial \theta}\right]^2 = \left[\frac{\partial}{\partial \theta}\theta\right]^2 = 1^2 = 1$$

To obtain the Fisher information $I(\theta)$, we have

$$\begin{aligned} I(\theta) &= -nE\left(\frac{\partial^2}{\partial \theta^2} \ln f_Y(Y)\right) = -nE\left(\frac{\partial^2}{\partial \theta^2} \ln\left(\frac{1}{\theta}\exp\left(-\frac{Y}{\theta}\right)\right)\right) \\ &= -nE\left(\frac{\partial^2}{\partial \theta^2}\left(-\ln\theta - \frac{Y}{\theta}\right)\right) = -nE\left(\frac{\partial}{\partial \theta}\left(\frac{Y}{\theta^2} - \frac{1}{\theta}\right)\right) = -nE\left(-\frac{2Y}{\theta^3} + \frac{1}{\theta^2}\right) \\ &= -n\left(-\frac{2E(Y)}{\theta^3} + \frac{1}{\theta^2}\right) = -n\left(-\frac{2\theta}{\theta^3} + \frac{1}{\theta^2}\right) = -n\left(-\frac{2}{\theta^2} + \frac{1}{\theta^2}\right) = \frac{n}{\theta^2} \end{aligned}$$

Therefore,

$$Z_{MLE} = \frac{t(\hat{\theta}) - t(\theta)}{\sqrt{\left[\frac{\partial t(\theta)}{\partial \theta}\right]^2 \Big/ I(\theta)}} = \frac{\overline{Y} - \theta}{\sqrt{1/(n/\theta^2)}} = \frac{\overline{Y} - \theta}{\theta/\sqrt{n}}$$

(b) Since $\hat{\theta} = \overline{Y}$ is the MLE of θ, we have that $t(\hat{\theta}) = \hat{\theta}^2 = \overline{Y}^2$ is the MLE of $t(\theta) = \theta^2$, and hence

$$\left[\frac{\partial t(\theta)}{\partial \theta}\right]^2 = \left[\frac{\partial}{\partial \theta}\theta^2\right]^2 = [2\theta]^2 = 4\theta^2$$

Combining with $I(\theta)$ that we obtained from part (a), we get

$$Z_{MLE} = \frac{t(\hat{\theta}) - t(\theta)}{\sqrt{\left[\frac{\partial t(\theta)}{\partial \theta}\right]^2 \Big/ I(\theta)}} = \frac{\overline{Y}^2 - \theta^2}{\sqrt{4\theta^2/(n/\theta^2)}} = \frac{\overline{Y}^2 - \theta^2}{2\theta^2/\sqrt{n}}$$

20. **(5880*)** If $Y \sim$ Poisson(λ), then have seen that $\hat{\lambda} = \overline{Y}$ is the MLE of λ (see Problem 11), so that the MLE of $t(\lambda) = P(Y = 0) = e^{-\lambda}$ will be given by $t(\hat{\lambda}) = e^{-\hat{\lambda}} = e^{-\overline{Y}}$. Hence,

$$\left[\frac{\partial t(\lambda)}{\partial \lambda}\right]^2 = \left[\frac{\partial}{\partial \lambda}e^{-\lambda}\right]^2 = [-e^{-\lambda}]^2 = e^{-2\lambda}$$

The Fisher Information is given by

$$\begin{aligned} I(\lambda) &= -nE\left(\frac{\partial^2}{\partial\lambda^2}\ln p_Y(Y)\right) = -nE\left(\frac{\partial^2}{\partial\lambda^2}\ln\left[\frac{e^{-\lambda}\lambda^Y}{Y!}\right]\right) \\ &= -nE\left(\frac{\partial^2}{\partial\lambda^2}\left(-\lambda + Y\ln\lambda - \ln(Y!)\right)\right) = -nE\left(\frac{\partial}{\partial\lambda}\left(-1+\frac{Y}{\lambda}\right)\right) \\ &= -nE\left(\frac{-Y}{\lambda^2}\right) = \frac{n}{\lambda^2}E(Y) = \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda} \end{aligned}$$

so therefore,

$$Z_{MLE} = \frac{t(\hat{\lambda}) - t(\lambda)}{\sqrt{\left[\frac{\partial t(\lambda)}{\partial\lambda}\right]^2 \Big/ I(\lambda)}} = \frac{e^{-\overline{Y}} - e^{-\lambda}}{\sqrt{e^{-2\lambda}/(n/\lambda)}} = \frac{e^{-\overline{Y}} - e^{-\lambda}}{e^{-\lambda}\sqrt{\lambda/n}} = \frac{e^{-(\overline{Y}-\lambda)} - 1}{\sqrt{\lambda/n}}$$

* A distribution with a random variable Y with a parameter θ is said to belong to an *exponential family* if $f_Y(y)$ or $P(Y=y)$ can be written as

$$h(y)c(\theta)\exp(w(\theta)t(y))$$

If a random variable Y has a distribution with *multiple* parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$, then Y belongs to an exponential family if $f_Y(y)$ or $P(Y=y)$ can be written as

$$h(y)c(\boldsymbol{\theta})\exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(y)\right)$$

For example, if $Y \sim N(\mu, \sigma^2)$, then by letting $\boldsymbol{\theta} = (\mu, \sigma^2)$, we have

$$\begin{aligned} f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\mu^2}{2\sigma^2}\right)\exp\left(-\frac{y^2}{2\sigma^2} + \frac{\mu y}{\sigma^2}\right) \\ &= h(y)c(\boldsymbol{\theta})\exp(w_1(\boldsymbol{\theta})t_1(y) + w_2(\boldsymbol{\theta})t_2(y)), \quad -\infty < y < \infty \end{aligned}$$

where

$$h(y) = 1,\ c(\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\mu^2}{2\sigma^2}\right),\ w_1(\boldsymbol{\theta}) = \frac{-1}{2\sigma^2},\ t_1(y) = y^2,\ w_2(\boldsymbol{\theta}) = \frac{\mu}{\sigma^2},\ t_2(y) = y$$

You can verify that $Y \sim \text{Poisson}(\theta)$ and $Y \sim \text{Gamma}(\alpha, \beta)$ belong to an exponential family.