Homework 7 Solution

1. Since the rejection region is given by $RR = \{Y > 0.5\}$, and Y is the test statistic, we have that

$$\text{power}(\theta) = P(\text{reject } H_0, \text{ given } \theta) = P(Y > 0.5) = \int_{0.5}^{1} f_Y(y)\, dy = \int_{0.5}^{1} \theta y^{\theta-1}\, dy = 1 - (0.5)^{\theta}$$

Since $\theta > 0$, the power function will start at 0 and increase, approaching 1 as $\theta \to \infty$.

2. Let us first consider $H_0 : \theta = 1$ versus $H_A^* : \theta = c$, where $c > 1$. Then we apply Theorem 10.1 with $L(\theta) = f_Y(y) = \theta y^{\theta-1}$ and

$$\frac{L(\theta_0)}{L(\theta_A)} = \frac{L(1)}{L(c)} = \frac{1}{cy^{c-1}} < k$$

which implies that

$$y > \left(\frac{1}{ck}\right)^{1/(c-1)} = (ck)^{-1/(c-1)} = k^*$$

Hence, we can have the rejection region $RR = \{Y > k^*\}$. Now,

$$\alpha = P(\text{reject } H_0, H_0 \text{ True}) = P(Y > k^*, \theta = 1) = \int_{k^*}^{1} f_Y(y)\, dy = \int_{k^*}^{1} 1\, dy = 1 - k^*$$

so we can get $k^* = 1 - \alpha$, or $RR = \{Y > 1 - \alpha\}$. Here, the RR does not depend on any particular value of alternative hypothesis $H_A^* : \theta = c$, so the Theorem 10.1 is true for every $c > 1$, and therefore, we conclude that the test $H_0 : \theta = 1$ versus $H_A : \theta > 1$ with $RR = \{Y > 1 - \alpha\}$ is the UMP level-α test.

3. Let us first consider the hypotheses $H_0 : \sigma^2 = \sigma_0^2$ versus $H_A^* : \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_0^2$. Then since μ is known, the likelihood function $L(\sigma^2)$ is given by

$$L(\sigma^2) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right]$$

Applying Theorem 10.1, we obtain

$$\begin{aligned}
\frac{L(\sigma_0^2)}{L(\sigma_1^2)} &= \frac{\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n \exp\left[\frac{-1}{2\sigma_0^2}\sum_{i=1}^{n}(y_i - \mu)^2\right]}{\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^n \exp\left[\frac{-1}{2\sigma_1^2}\sum_{i=1}^{n}(y_i - \mu)^2\right]} \\
&= \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^n \exp\left[-\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right)\sum_{i=1}^{n}(y_i - \mu)^2\right] \\
&= \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^n \exp\left[-\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_1^2}\right)\frac{\sum_{i=1}^{n}(y_i - \mu)^2}{\sigma_0^2}\right] < k
\end{aligned}$$

From the hint, we note that $\sum_{i=1}^n (Y_i-\mu)^2/\sigma^2 = \sum_{i=1}^n [(Y_i-\mu)/\sigma]^2 \sim \chi^2(n)$, and under $H_0: \sigma^2 = \sigma_0^2$, we get $\sum_{i=1}^n (Y_i-\mu)^2/\sigma_0^2 \sim \chi^2(n)$, which we can isolate from above inequality, i.e., after taking the ln,

$$\frac{\sum_{i=1}^n (y_i-\mu)^2}{\sigma_0^2} > \left(n \ln\left(\frac{\sigma_1^2}{\sigma_0^2}\right) - \ln k \right) \frac{2\sigma_1^2}{(\sigma_1^2-\sigma_0^2)} = k'$$

so that $RR = \{\sum_{i=1}^n (Y_i-\mu)^2/\sigma_0^2 > k'\}$ Now, again, using the hint,

$$\alpha = P(\text{reject } H_0,\ H_0 \text{ True}) = P\left(\frac{\sum_{i=1}^n (Y_i-\mu)^2}{\sigma_0^2} > k'\right) = P\left(\frac{\sum_{i=1}^n (Y_i-\mu)^2}{\sigma_0^2} > \chi^2_{\alpha,n}\right)$$

from which we obtain $k' = \chi^2_{\alpha,n}$, and so

$$RR = \left\{ \frac{\sum_{i=1}^n (Y_i-\mu)^2}{\sigma_0^2} > \chi^2_{\alpha,n} \right\}$$

Since the RR does not depend on any particular value of alternative hypothesis σ_1^2, the Theorem 10.1 is true for every $\sigma_1^2 > \sigma_0^2$, and therefore, it is the UMP level-α for $H_A: \sigma^2 > \sigma_0^2$.

4. Since $P(Y=y) = \theta^y(1-\theta)^{1-y}$, $y = 0, 1$, and we have seen that

$$L(\theta) = \theta^{\sum_{i=1}^n y_i}(1-\theta)^{n-\sum_{i=1}^n y_i}$$

with $\hat{\theta}_{MLE} = \overline{Y}$, we obtain, for $H_0: \theta = \theta_0$ versus $H_A: \theta \neq \theta_0$,

$$\begin{aligned}
\lambda &= \frac{L(\theta_0)}{L(\hat{\theta}_{MLE})} = \frac{\theta_0^{\sum_{i=1}^n y_i}(1-\theta_0)^{n-\sum_{i=1}^n y_i}}{\hat{\theta}_{MLE}^{\sum_{i=1}^n y_i}(1-\hat{\theta}_{MLE})^{n-\sum_{i=1}^n y_i}} = \frac{\theta_0^{\sum_{i=1}^n y_i}(1-\theta_0)^{n-\sum_{i=1}^n y_i}}{\overline{y}^{\sum_{i=1}^n y_i}(1-\overline{y})^{n-\sum_{i=1}^n y_i}} \\
&= \left(\frac{\theta_0}{\overline{y}}\right)^{\sum_{i=1}^n y_i} \left(\frac{1-\theta_0}{1-\overline{y}}\right)^{n-\sum_{i=1}^n y_i} = \left(\frac{\theta_0}{\overline{y}}\right)^{n\overline{y}} \left(\frac{1-\theta_0}{1-\overline{y}}\right)^{n-n\overline{y}} = \left(\frac{\theta_0}{\overline{y}}\right)^{n\overline{y}} \left(\frac{1-\theta_0}{1-\overline{y}}\right)^{n(1-\overline{y})} \\
&= \left[\left(\frac{\theta_0}{\overline{y}}\right)^{\overline{y}} \left(\frac{1-\theta_0}{1-\overline{y}}\right)^{1-\overline{y}} \right]^n
\end{aligned}$$

5. Since $P(Y=y) = \frac{\theta^y e^{-\theta}}{y!}$, $y = 0, 1, \ldots$, and we have seen that

$$L(\theta) = \frac{\theta^{\sum_{i=1}^n y_i} e^{-n\theta}}{\prod_{i=1}^n y_i}$$

with $\hat{\theta}_{MLE} = \overline{Y}$, we obtain, for $H_0: \theta = \theta_0$ versus $H_A: \theta \neq \theta_0$,

$$\begin{aligned}
\lambda &= \frac{L(\theta_0)}{L(\hat{\theta}_{MLE})} = \frac{\theta_0^{\sum_{i=1}^n y_i} e^{-n\theta_0}/\prod_{i=1}^n y_i}{\hat{\theta}_{MLE}^{\sum_{i=1}^n y_i} e^{-n\hat{\theta}_{MLE}}/\prod_{i=1}^n y_i} = \frac{\theta_0^{\sum_{i=1}^n y_i} e^{-n\theta_0}}{\overline{y}^{\sum_{i=1}^n y_i} e^{-n\overline{y}}} \\
&= \left(\frac{\theta_0}{\overline{y}}\right)^{\sum_{i=1}^n y_i} e^{-n(\theta_0-\overline{y})} = \left(\frac{\theta_0}{\overline{y}}\right)^{n\overline{y}} e^{-n(\theta_0-\overline{y})} \\
&= \left[\left(\frac{\theta_0}{\overline{y}}\right)^{\overline{y}} e^{-(\theta_0-\overline{y})} \right]^n
\end{aligned}$$

6. If $Y_1, \ldots, Y_n \sim$ iid $N(\theta, 1)$, we know, with $\mu = \theta$ and $\sigma^2 = 1$, that

$$L(\theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(y_i - \theta)^2\right]$$

and $\hat{\theta}_{MLE} = \overline{Y}$. Then for $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$, we obtain

$$\begin{aligned}
\lambda &= \frac{L(\theta_0)}{L(\hat{\theta}_{MLE})} = \frac{(1/\sqrt{2\pi})^n \exp[-\sum_{i=1}^{n}(y_i - \theta_0)^2/2]}{(1/\sqrt{2\pi})^n \exp[-\sum_{i=1}^{n}(y_i - \hat{\theta}_{MLE})^2/2]} \\
&= \frac{\exp[-\sum_{i=1}^{n}(y_i - \theta_0)^2/2]}{\exp[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2]} = \frac{\exp[-\sum_{i=1}^{n}(y_i - \overline{y} + \overline{y} - \theta_0)^2/2]}{\exp[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2]} \\
&= \frac{\exp[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2 - \sum_{i=1}^{n}(\overline{y} - \theta_0)^2/2 + \sum_{i=1}^{n}(y_i - \overline{y})(\overline{y} - \theta_0)]}{\exp[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2]} \\
&= \frac{\exp[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2 - n(\overline{y} - \theta_0)^2/2 + 0]}{\exp[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2]} \\
&= \exp\left[-\sum_{i=1}^{n}(y_i - \overline{y})^2/2 - n(\overline{y} - \theta_0)^2/2 + \sum_{i=1}^{n}(y_i - \overline{y})^2/2\right] \\
&= \exp\left[\frac{-n(\overline{y} - \theta_0)^2}{2}\right]
\end{aligned}$$