Homework 8 Solution

1. We are testing
$$H_0 : \mu = 130 \quad \text{versus} \quad H_A : \mu < 130$$
at level $\alpha = 0.05$. So we have that $\mu_0 = 130$, along with given numbers given by $\overline{Y} = 128.6$ and $s = 2.1$ and $n = 40$. We can assume that $\sigma = 2.1$ since n is sufficiently large. Then, our test statistic is
$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{128.6 - 130}{2.1/\sqrt{40}} = -4.22$$
and comparing against the critical value $-z_\alpha = -z_{0.05} = -1.645$, we see that $Z < -z_\alpha$, which is in the rejection region $RR = \{Z < -z_\alpha\}$. Therefore, we reject H_0.

2. We are testing
$$H_0 : \mu = 5 \quad \text{versus} \quad H_A : \mu > 5$$
at level $\alpha = 0.05$. Then $\mu_0 = 5$, and we have $\overline{Y} = 5.4$ and $s^2 = 9.61$ and $n = 500$. We can assume that $\sigma = \sqrt{\sigma^2} = \sqrt{9.61} = 3.1$ since n is large. Out test statistic is
$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{5.4 - 5}{3.1/\sqrt{500}} = 2.89$$
and comparing against the critical value $z_\alpha = z_{0.05} = 1.645$, we see that $Z > z_\alpha$, which is in the rejection region $RR = \{Z > z_\alpha\}$. Therefore, we reject H_0.

3. Here, $n = 15$ and $\overline{Y} = 33,800$, and $\sigma = 4000$ is known. If testing $H_0 : \mu = 32,500$ versus $H_1 : \mu > 32,500$ at level $\alpha = 0.05$, then we determine that $\mu_0 = 32,500$ and $z_\alpha = z_{0.05} = 1.645$, and
$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{33,800 - 32,500}{4000/\sqrt{15}} = 1.26$$
Since $Z < z_\alpha$, we "accept" H_0 here.

4. We have that $n = 30$, $\sigma = 0.4$, and we compute that $\overline{Y} = 12.76$. Note that we test for $H_0 : \mu = 12.6$ versus $H_1 : \mu \neq 12.6$ at level $\alpha = 0.05$. Then $\mu_0 = 1.26$ and $z_{\alpha/2} = z_{0.025} = 1.96$. and
$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} = \frac{12.76 - 12.6}{0.4/\sqrt{30}} = 2.19$$
so that $|Z| = 2.19 > 1.96 = z_{\alpha/2}$. Hence, H_0 is rejected.

5. Starting with the hypotheses $H_0 : \mu = \mu_0$ versus $H_A : \mu < \mu_0$, and β is evaluated at $\mu = \mu_A$ where $\mu_A < \mu_0$, we have that the rejection region is $RR = \{\overline{Y} < k\}$. Hence
$$\begin{aligned} \alpha &= P(\text{reject } H_0, \text{ when } H_0 \text{ true}) = P(\overline{Y} < k, \ \ \mu = \mu_0) = P\left(\frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} < \frac{k - \mu}{\sigma/\sqrt{n}}, \ \ \mu = \mu_0\right) \\ &= P\left(Z < \frac{k - \mu_0}{\sigma/\sqrt{n}}\right) = P(Z < -z_\alpha) \end{aligned}$$

and

$$\begin{aligned}\beta &= P(\text{accept } H_0, \text{ when } H_A \text{ true}) = P(\overline{Y} \geq k, \ \ \mu = \mu_A) = P\left(\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}} \geq \frac{k-\mu}{\sigma/\sqrt{n}}, \ \ \mu = \mu_A\right) \\ &= P\left(Z \geq \frac{k-\mu_A}{\sigma/\sqrt{n}}\right) = P(Z \geq z_\beta)\end{aligned}$$

(why?) so that

$$\frac{k-\mu_0}{\sigma/\sqrt{n}} = -z_\alpha \quad \text{and} \quad \frac{k-\mu_A}{\sigma/\sqrt{n}} = z_\beta$$

Then

$$k = \mu_0 - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) \quad \text{and} \quad k = \mu_A + z_\beta\left(\frac{\sigma}{\sqrt{n}}\right)$$

which then follows that

$$\mu_0 - z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right) = \mu_A + z_\beta\left(\frac{\sigma}{\sqrt{n}}\right) \implies \mu_0 - \mu_A = (z_\alpha + z_\beta)\left(\frac{\sigma}{\sqrt{n}}\right)$$

which implies that

$$\sqrt{n} = \frac{(z_\alpha + z_\beta)\sigma}{\mu_0 - \mu_A}$$

Therefore,

$$n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_0 - \mu_A)^2} = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_A - \mu_0)^2}$$

as was to be seen.

6. First, use the rejection region $RR = \{Z < -z_\alpha\}$ to see that, under H_0 and $\alpha = 0.01$, we have that $z_\alpha = z_{0.01} = 2.326$ and that

$$Z = \frac{\overline{Y}-\mu_0}{\sigma/\sqrt{n}} < -2.326$$

and since $\mu_0 = 130$ and $\sigma = 2.1$ and $n = 40$, we have that

$$\frac{\overline{Y}-\mu_0}{\sigma/\sqrt{n}} < -2.326 \implies \overline{Y} - \mu_0 < -2.326\left(\frac{\sigma}{\sqrt{n}}\right)$$

which implies that

$$\overline{Y} < \mu_0 - 2.326\left(\frac{\sigma}{\sqrt{n}}\right) = 130 - 2.326\left(\frac{2.1}{\sqrt{40}}\right) = 129.2277$$

so that $k = 129.2277$ for the rejection region $RR = \{\overline{Y} < k\}$. Now, since $\mu_A = 128$, we can compute

$$\begin{aligned}\beta &= P(\text{accept } H_0, \text{ when } H_A \text{ true}) = P(\overline{Y} \geq k, \ \ \mu = \mu_A) = P\left(\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}} \geq \frac{k-\mu}{\sigma/\sqrt{n}}, \ \ \mu = \mu_A\right) \\ &= P\left(Z \geq \frac{k-\mu_A}{\sigma/\sqrt{n}}\right) = P\left(Z \geq \frac{129.45-128}{2.1/\sqrt{40}}\right) = P(Z > 3.697456) \\ &\approx 0\end{aligned}$$

7. (a) Use the rejection region $RR = \{Z > z_\alpha\}$ to see that, under H_0 and $\alpha = 0.05$, we have that $z_\alpha = z_{0.05} = 1.645$ and that

$$Z = \frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} > 1.645$$

and since $\mu_0 = 5$ and $\sigma = 3.1$ and $n = 500$, we have that

$$\frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} > 1.645 \implies \overline{Y} > \mu_0 + 1.645\left(\frac{\sigma}{\sqrt{n}}\right) = 5 + 1.645\left(\frac{3.1}{\sqrt{500}}\right) = 5.228$$

so that $k = 5.228$ for the rejection region $RR = \{\overline{Y} > k\}$. Since $\mu_A = 5.5$, we see that

$$\begin{aligned}
\beta &= P(\text{accept } H_0, \text{ when } H_A \text{ true}) = P(\overline{Y} \leq k, \ \ \mu = \mu_A) = P\left(\frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \leq \frac{k - \mu}{\sigma/\sqrt{n}}, \ \ \mu = \mu_A\right) \\
&= P\left(Z \leq \frac{k - \mu_A}{\sigma/\sqrt{n}}\right) = P\left(Z \leq \frac{5.228 - 5.5}{3.1/\sqrt{500}}\right) = P(Z \leq -1.96) = P(Z > 1.96) \\
&= 0.025
\end{aligned}$$

(b) Since $\alpha = 0.01$ and $\beta = 0.05$, we have that $z_\alpha = z_{0.01} = 2.33$ and $z_\beta = z_{0.05} = 1.645$. Then we substitute the numbers from part (a) to see that

$$n = \frac{(z_\alpha + z_\beta)^2\sigma^2}{(\mu_A - \mu_0)^2} = \frac{(2.33 + 1.645)^2(3.1)^2}{(5.5 - 5)^2} = 607.376$$

Hence, we need

$$n = 608$$

to ensure that $\alpha = 0.01$ and $\beta = 0.05$ when $\mu_A = 5.5$.