Quiz 2
September 15, 2026

1. Let Y_1 and Y_2 have the joint pdf

$$f_{Y_1,Y_2}(y_1,y_2)=\frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2},\quad -\infty<y_1<\infty \text{ and } -\infty<y_2<\infty.$$

Suppose that $U_1=Y_1+Y_2$ and $U_2=Y_1-Y_2$. Find the joint pdf of (U_1,U_2).

(Solution)

Suppose that $U_1=Y_1+Y_2$ and $U_2=Y_1-Y_2$. Then

$$Y_1=\frac{U_1+U_2}{2}\quad\text{and}\quad Y_2=\frac{U_1-U_2}{2}$$

so that

$$J=\begin{vmatrix}\frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2}\\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2}\end{vmatrix}=\begin{vmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{1}{2}$$

Hence,

$$\begin{aligned}f_{U_1,U_2}(u_1,u_2) &= f_{Y_1,Y_2}(y_1,y_2)|J|=\frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}\left|-\frac{1}{2}\right|\\ &= \frac{1}{4\pi}e^{-((u_1+u_2)^2+(u_1-u_2)^2)/8}=\frac{1}{4\pi}e^{-(u_1^2+u_2^2)/4}\\ &= \frac{1}{\sqrt{4\pi}}e^{-u_1^2/4}\frac{1}{\sqrt{4\pi}}e^{-u_2^2/4}\end{aligned}$$

for $-\infty<u_1<\infty$ and $-\infty<u_2<\infty$.

(It turns out that we start with independent $Y_1\sim N(0,1)$ and $Y_2\sim N(0,1)$, and obtain $U_1\sim N(0,2)$ and $U_2\sim N(0,2)$, which are also independent).