Chapter 2 Solution

2.8 (a) Since $E(Y|X=x) = \beta_0 + \beta_1 x$, then $\alpha = \beta_0 + \beta_1\bar{x} = E(Y|X=\bar{x})$.

(b) Since $RSS(\alpha, \beta_1) = \sum(y_i - \alpha - \beta_1(x_i - \bar{x}))^2$, we have that

$$\begin{aligned}
\frac{\partial}{\partial\alpha} RSS(\alpha, \beta_1) &= -2\sum(y_i - \alpha - \beta_1(x_i - \bar{x})) = 0 \\
\frac{\partial}{\partial\beta_1} RSS(\alpha, \beta_1) &= -2\sum(y_i - \alpha - \beta_1(x_i - \bar{x}))(x_i - \bar{x}) = 0
\end{aligned}$$

where

$$\begin{aligned}
& \sum(y_i - \alpha - \beta_1(x_i - \bar{x})) = \sum y_i - n\alpha - \beta_1\sum(x_i - \bar{x}) = \sum y_i - n\alpha = 0 \\
\Longrightarrow\quad & \hat{\alpha} = \bar{y}
\end{aligned}$$

and

$$\sum(y_i - \alpha - \beta_1(x_i - \bar{x}))(x_i - \bar{x}) = \sum(y_i(x_i - \bar{x}) - \alpha(x_i - \bar{x}) - \beta_1(x_i - \bar{x})^2) = 0$$

so that upon replacing α by $\hat{\alpha} = \bar{y}$, we obtain

$$\begin{aligned}
& \sum(y_i(x_i - \bar{x}) - \hat{\alpha}(x_i - \bar{x}) - \beta_1(x_i - \bar{x})^2) = \sum(y_i(x_i - \bar{x}) - \bar{y}(x_i - \bar{x}) - \beta_1(x_i - \bar{x})^2) = 0 \\
\Longrightarrow\quad & \sum(x_i - \bar{x})(y_i - \bar{y}) - \beta_1\sum(x_i - \bar{x})^2 = 0 \\
\Longrightarrow\quad & \sum(x_i - \bar{x})(y_i - \bar{y}) = \beta_1\sum(x_i - \bar{x})^2 \\
\Longrightarrow\quad & \hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} = \frac{SXY}{SXX}
\end{aligned}$$

(c) Since $\mathrm{Var}(y_i) = \sigma^2$, we have that

$$\mathrm{Var}(\hat{\alpha}) = \mathrm{Var}(\bar{y}) = \frac{\sigma^2}{n}$$

For β_1, it is the same as OLS,

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{SXX}$$

Now, the covariance between them is $\mathrm{Cov}(\hat{\alpha}, \hat{\beta}_1) = \mathrm{Cov}(\bar{y}, \hat{\beta}_1)$, which is found to be 0 (see Appendix), or we can derive (with $\hat{\beta}_1 = \sum c_i y_i$ where $c_i = (x_i - \bar{x})/SXX$ and $\sum c_i = 0$)

$$\begin{aligned}
\mathrm{Cov}(\bar{y}, \hat{\beta}_1) &= \mathrm{Cov}\left(\frac{1}{n}\sum y_i, \sum c_i y_i\right) = \frac{1}{n}\sum c_i \mathrm{Cov}(y_i, y_i) \\
&= \frac{1}{n}\sum c_i \mathrm{Var}(y_i) = \frac{\sigma^2}{n}\sum c_i = 0
\end{aligned}$$

2.12 We have

$$r_{xy} = \frac{s_{xy}}{SD_x SD_y} = \frac{SXY/(n-1)}{\sqrt{SXX/(n-1)}\sqrt{SYY/(n-1)}} = \frac{SXY}{\sqrt{SXX}\sqrt{SYY}}$$

and

$$\sqrt{1-r_{xy}^2} = \sqrt{1 - \frac{SXY^2}{SXX \cdot SYY}} = \sqrt{\frac{SXX \cdot SYY - SXY^2}{SXX \cdot SYY}} = \frac{\sqrt{SXX \cdot SYY - SXY^2}}{\sqrt{SXX}\sqrt{SYY}}$$

so that (working backwards)

$$\begin{aligned}
\sqrt{n-2}\frac{r_{xy}}{\sqrt{1-r_{xy}^2}} &= \sqrt{n-2}\frac{SXY/(\sqrt{SXX}\sqrt{SYY})}{\sqrt{(SXX \cdot SYY - SXY^2)}/(\sqrt{SXX}\sqrt{SYY})} \\
&= \sqrt{n-2}\frac{SXY}{\sqrt{(SXX \cdot SYY - SXY^2)}} \\
&= \sqrt{n-2}\frac{SXY}{\sqrt{RSS \cdot SXX}} \qquad \left(\text{since } RSS = SYY - \frac{SXY^2}{SXX}\right)
\end{aligned}$$

whereas

$$\frac{\hat{\beta}_1}{\text{se}(\hat{\beta}_1)} = \frac{SXY/SXX}{\hat{\sigma}/\sqrt{SXX}} = \frac{SXY/\sqrt{SXX}}{\sqrt{RSS/(n-2)}} = \sqrt{n-2}\frac{SXY}{\sqrt{RSS \cdot SXX}}$$

from which we conclude that

$$t = \frac{\hat{\beta}_1}{\text{se}(\hat{\beta}_1)} = \sqrt{n-2}\frac{r_{xy}}{\sqrt{1-r_{xy}^2}}$$

2.18 Here, we have that

$$\begin{aligned}
RSS &= \sum \hat{e}_i^2 = \sum (y_i - \hat{y}_i)^2 = \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = \sum (y_i - \bar{y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i)^2 \\
&= \sum ((y_i - \bar{y}) + \hat{\beta}_1 (x_i - \bar{x}))^2 = \sum [(y_i - \bar{y})^2 + \hat{\beta}_1^2 (x_i - \bar{x})^2 - 2(y_i - \bar{y})\hat{\beta}_1 (x_i - \bar{x})] \\
&= \sum (y_i - \bar{y})^2 + \hat{\beta}_1^2 \sum (x_i - \bar{x})^2 - 2\hat{\beta}_1 \sum (y_i - \bar{y})(x_i - \bar{x}) \\
&= SYY + \hat{\beta}_1^2 SXX - 2\hat{\beta}_1 SXY = SYY + \left(\frac{SXY}{SXX}\right)^2 SXX - 2\left(\frac{SXY}{SXX}\right) SXY \\
&= SYY + \frac{(SXY)^2}{SXX} - 2\frac{(SXY)^2}{SXX} \\
&= SYY - \frac{(SXY)^2}{SXX} = SYY - \hat{\beta}_1^2 SXX
\end{aligned}$$