Midterm Exam Solution

1. Since $RSS(\beta) = \sum(y_i - \beta x_i)^2$,

$$\frac{d}{d\beta}RSS(\beta) = \frac{d}{d\beta}\sum(y_i - \beta x_i)^2 = -2\sum(y_i - \beta x_i)x_i = 0$$
$$\implies \sum x_i y_i - \beta\sum x_i^2 = 0 \implies \sum x_i y_i = \beta\sum x_i^2$$
$$\implies \hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$$

You may also solve this problem by using the formula $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ with $\mathbf{X} = (x_1, \ldots, x_n)'$ and $\mathbf{Y} = (y_1, \ldots, y_n)'$, so that

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} = \left(\sum x_i^2\right)^{-1}\sum x_i y_i = \frac{\sum x_i y_i}{\sum x_i^2}$$

2. (a) Note that $n = 74$ and $p = 5$. We need

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1> X<-matrix(c(rep(1,74),x1,x2,x3,x4,x5),ncol=6)
6> sigma2hat<-rss/(68)
```

so the answers are $6 (= p + 1)$ and $68 (= n - p - 1)$.

(b) We want

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pt(abs(tstat), 68, lower.tail=F)*2
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3. First, since $\widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$ (see also Problem 2), we compute

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Then, we compute

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

from whch we obtain (with $\hat{\sigma}^2 = 12$)

$$\widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1} = 12\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/6 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Now, $\mathrm{se}(\hat{\beta}_0)$ is simply a square root of the $[1,1]$ element of $\widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}})$, which is $\sqrt{4} = 2$, so that we have

$$\mathrm{se}(\hat{\beta}_0) = 2$$