Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 1. Due Friday, February 2.

1. Prove or disprove convergence for each of the following series (p, q are real parameters and convergence may depend on their values).

$$\text{(i)} \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}, \qquad \text{(ii)} \sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\log n}, \qquad \text{(iii)} \sum_{n=1}^{\infty} \left(\left(1 + \frac{1}{n}\right)^n - e \right)^{\sqrt{2}},$$

$$\text{(iv)} \sum_{n=2}^{\infty} n^p \log(n)^q, \text{(v)} \sum_{n=1}^{\infty} \frac{1}{n(n^{1/n})^{99}}$$

2. Prove or disprove convergence for each of the following sequences and in case of convergence, determine the limit:
(i) $a_n = \sqrt{n^4 + \cos(n^2)} - n^2$
(ii) $b_n = n^2 + \frac{1}{2}n - \sqrt{n^4 + n^3}$
(iii) $c_n = \sum_{k=n}^{n^2} \frac{1}{k}$
(iv) $d_n = n \sum_{k=0}^{\infty} \frac{1}{n^2+k^2}$

3. Let X be a metric space and $(f_n)_n$ a sequence of continuous functions $f_n : X \to \mathbb{R}$ that converges uniformly to a function $f : X \to \mathbb{R}$.
(i) Show that for every sequence $(x_n)_x \subset X$ converging to some $x \in X$ we have

$$\lim_{n\to\infty} f_n(x_n) = f(x).$$

(ii) Is the converse true? That is, is it true that if $\lim_{n\to\infty} f_n(x_n) = x$ holds for every $x_n \to x$ that $f_n \to f$ uniformly? Give a proof or counterexample.

4. For a positive real number x define

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n(n+1) + x}.$$

(i) Show that $f : (0, \infty) \to (0, \infty)$ is a well-defined and continuous function.
(ii) Prove that there exists a unique $x_0 \in (0, \infty)$ such that $f(x_0) = 2\pi$.
(iii) Determine the value of x_0. *Hint:* Use the Leibniz formula for π.

5. For which $x \in \mathbb{R}$ do the following series converge? On which sets do these series converge uniformly?

$$\text{(i)} \sum_{n=1}^{\infty} n^{10} x^n, \qquad \text{(ii)} \sum_{n=1}^{\infty} (2^{1/n} - 1)^n x^n \qquad \text{(iii)} \sum_{n=1}^{\infty} \tan(n^{-2}) e^{nx}.$$