Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 10. Due Monday, April 30.

(Problems with an asterisk (*) are optional.)

1. Let $\mathcal{D} \subset \mathbb{R}^2$ be a finite set. Define a function $E : \mathbb{R}^3 \to \mathbb{R}$ by

$$E(a, b, c) = \sum_{x \in \mathcal{D}} (ax_1^2 + bx_1 + c - x_2)^2.$$

(1) Show that E is convex.
(2) Does there exist a set $\mathcal{D}$ such that E is strongly convex? Proof or counterexample.

2. (a) Find a convex function that is not bounded from below.
(b) Find a strictly convex function that is not bounded from below.
(c) If a function is strictly convex and bounded from below, does it necessarily have a critical point? (Proof or counterexample.)
(d) Is every convex function continuous? (Proof or counterexample.)

3. Construct a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at x for every $x \in \mathbb{Q}$.

4. Let $f \in C^2(\mathbb{R}^n)$. Recall that we defined f to be *strongly convex* if there exists $\beta > 0$ such that $\langle D^2 f|_x y, y\rangle \geq \beta \|y\|^2$ for every $x, y \in \mathbb{R}^n$. Show that f is strongly convex if and only if there exists $\gamma > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \gamma t(1-t)\|x - y\|^2$$

for all $x, y \in \mathbb{R}^n, t \in [0, 1]$.
(Consequently, that condition can serve as an alternative definition of strong convexity, which is also valid if f is not C^2.)

5*. Let $K \subset \mathbb{R}^n$ be compact. Suppose that $f \in C^2(\mathbb{R}^n)$ is strictly convex. Show that there exist $\beta_-, \beta_+ > 0$ such that

$$\beta_- \|y\|^2 \leq \langle D^2 f|_x y, y\rangle \leq \beta_+ \|y\|^2$$

for all $x \in K$ and $y \in \mathbb{R}^n$. *Hint:* Consider the minimal eigenvalue of $D^2 f|_x$ as a function of x.

Turn the page.

6*. Fix a function $\sigma \in C^1(\mathbb{R})$ and define for $x \in \mathbb{R}^n, W \in \mathbb{R}^{m\times n}, v \in \mathbb{R}^m$,

$$\mu(x, W, v) = \sum_{i=1}^{m} \sigma((Wx)_i)v_i$$

Given a finite set of points $\mathcal{D} = \{(x_1, y_1), \ldots, (x_N, y_N) \in \mathbb{R}^n \times \mathbb{R}\}$ define

$$E(W, v) = \sum_{j=1}^{N} (\mu(x_i, W, v) - y_i)^2.$$

Is E necessarily convex? (Proof or counterexample.)