Math 522, Spring 2018 – Analysis II
Homework assignment 2. Due Monday, February 12.

1. (i) Evaluate $\sum_{k=1}^{\infty} k^3 x^k$ for $|x| < 1$.
(ii) Evaluate $\sum_{k=2}^{\infty} k^{-1} x^{2k+1}$ for $|x| < 1$. What can you say for $x = 1$ and $x = -1$?

2. Let
$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$
Show that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.
Hint: Prove by induction on n that for $x > 0$, $f^{(n)}(x) = P_n(1/x)e^{-1/x}$ where P_n is a polynomial.

3. Let f be as in Problem 2.
(i) Does the Taylor series expanded at 0 represent the function $f(x)$ in an open interval containing 0?
(ii) Discuss the validity (true or false?) of the following statement: There is an open interval $(-r, r)$ and a positive constant A so that for every n the inequality $\sup_{x\in(-r,r)} |f^{(n)}(x)| \le A^n n!$ holds.
(iii) Let $g(x) = f(1 - x^2)$. Show that g is a C^∞-function on $\mathbb{R}$ with the property that $g(x) > 0$ for $x \in (-1, 1)$ and $g(x) = 0$ for $x \notin (-1, 1)$.
(iv) Given an interval (a, b) construct a C^∞-function h on $\mathbb{R}$ with the property that $h(x) > 0$ for $x \in (a, b)$ and $h(x) = 0$ for $x \notin (a, b)$.

4. (i) Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real or complex numbers with limit L. Prove that
$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = L$$
Definition: Given the sequence a_k, form the partial sums $s_n = \sum_{k=1}^{n} a_k$ and let
$$\sigma_N = \frac{s_1 + \cdots + s_N}{N}.$$
σ_N is called the Nth Cesàro mean of the sequence s_k or the Nth Cesàro sum of the series $\sum_{k=1}^{\infty} a_k$. If σ_N converges to a limit S we say that the series $\sum_{k=1}^{\infty} a_k$ is Cesàro summable to S.

Turn the page.

4. cont. (ii) Prove that if $\sum_{k=1}^{\infty} a_k$ is summable to S (i.e. by definition converges with sum S) then $\sum_{k=1}^{\infty} a_k$ is Cesàro summable to S.

(iii) Prove that the sum $\sum_{k=1}^{\infty}(-1)^{k-1}$ does not converge but is Cesàro summable to some limit S and determine S.

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. The series $\sum_{k=1}^{\infty} a_k$ is called *Abel summable* to S if for every r with $0 \le r < 1$ the series

$$A(r) = \sum_{k=1}^{\infty} a_k r^k$$

converges and if

$$\lim_{r \to 1-} A(r) = S\,.$$

Consider the series $1-2+3-4+5-6+-... = \sum_{k=1}^{\infty}(-1)^{k-1}k$ which of course is not convergent.

(i) Show that this series is Abel summable to some S and determine S.

(ii) Show that this series is not Cesàro summable to any number.

(*Hint:* If $\sum_{k=1}^{\infty} c_k$ is Cesàro summable then verify that c_n/n tends to 0).

Honors problem 1 (not graded):

Abel's theorem tells us that if $\sum_{k=1}^{\infty} a_k$ is summable to S then the series is also Abel summable to S. This exercise strengthens this implication to

$$\sum_{k=1}^{\infty} a_k \text{ summable} \implies \sum_{k=1}^{\infty} a_k \text{ Cesàro summable} \implies \sum_{k=1}^{\infty} a_k \text{ Abel summable}$$

where the first implication is the result of No. 4(ii). We are concerned with the second implication:

(i) Prove first that if $\sum_k a_k$ is Cesàro summable then for $r < 1$ we have

$$\sum_{k=1}^{\infty} a_k r^k = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n$$

(ii) Split the right hand side of the previous formula as

$$(1-r)^2 \sum_{n=1}^{\infty} n(\sigma_n - S) r^n + (1-r)^2 \sum_{n=1}^{\infty} nSr^n$$

and compute the second term. Then take the limit of the first term as $r \to 1-$ and finish the proof of the second implication.