Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 3. Due Monday, February 19.

(Problems with an asterisk (*) are optional.)

1. Recall the Fejér kernel $K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} e^{ikx}$ for integer $N \geq 0$.
(i) Show that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = 1.$$
(ii) Prove that
$$K_N(x) = \frac{1}{N+1}\frac{1-\cos((N+1)x)}{1-\cos(x)}.$$
(iii) For all $\pi \geq \delta > 0$ prove that
$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\delta}^{\delta} K_N(x)dx = 1.$$
(Thus, $(K_N)_N$ is an approximation of unity.)

2. Let f be the 2π-periodic function satisfying $f(x) = x$ for $x \in [-\pi, \pi)$.
(i) Compute the Fourier coefficients
$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-itn}dt. \ (n \in \mathbb{Z})$$
(ii) Use Parseval's theorem to conclude that
$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. Show that there exists no continuous 2π-periodic function g such that $f * g = f$ holds for all continuous 2π-periodic functions f.
Hint: Use the Riemann-Lebesgue lemma.

4. Define a sequence of polynomials $(T_n)_n$ by $T_0(x) = 1$, $T_1(x) = x$ and the following recurrence relation:
$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$
for $n \geq 2$.
(i) Show that $T_n(x) = \cos(nt)$ if $x = \cos(t)$.
Hint: Use that $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$ for all $a, b \in \mathbb{C}$.
(ii) Compute
$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}}$$
for all non-negative integers n, m.

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4. cont. (iii) Prove that $|T_n(x)| \leq 1$ for $x \in [-1,1]$ and determine when there is equality.

5*. Construct a sequence of real-valued polynomials $(p_n)_n$ such that

$$\int_0^1 p_n(x)p_m(x)dx = \begin{cases} 1, & \text{if}\, n = m, \\ 0, & \text{if}\, n \neq m \end{cases}$$

for all non-negative integers n, m.

6*. A metric space is called *separable* if it has a countable dense subset. Show that every compact metric space is separable.
Hint: For every positive integer n consider an open cover formed by open $1/n$-balls.