Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 4. Due Monday, February 26.

(Problems with an asterisk (*) are optional.)

1. Let f be the 2π-periodic function such that $f(x) = |x|$ for $x \in [-\pi, \pi]$. Determine explicitly a sequence of trigonometric polynomials $(p_N)_N$ such that $p_N \to f$ uniformly as $N \to \infty$.

2. Let $f \in C([0, 1])$ and $\mathcal{A} \subset C([0, 1])$ dense. Suppose that

$$\int_0^1 f(x)\overline{a(x)}dx = 0$$

for all $a \in \mathcal{A}$. Show that $f = 0$. *Hint:* Show that $\int_0^1 |f(x)|^2 dx = 0$.

3. Let $f \in C([-1, 1])$ and $a \in [-1, 1]$. Show that for every $\varepsilon > 0$ there exists a polynomial p such that $p(a) = f(a)$ and $|f(x) - p(x)| < \varepsilon$ for all $x \in [-1, 1]$.

4. Prove that

$$-\frac{1}{2} = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(n)}{n}.$$

Hint: Use a theorem on convergence of Fourier series.

5. Suppose $f \in C([1, \infty))$ and $\lim_{x \to +\infty} f(x) = a$. Show that f can be uniformly approximated on $[1, \infty)$ by functions of the form $g(x) = p(1/x)$, where p is a polynomial.