Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 5. Due Monday, March 5.

(Problems with an asterisk (*) are optional.)

1. Given a Cauchy sequence in a metric space which has a convergent subsequence, show that the whole sequence converges.

2. Let (X, d) be a metric space and $A \subset X$ a subset.
(i) Show that A is totally bounded if and only if $\overline{A}$ is totally bounded.
(ii) Assume that X is complete. Show that A is totally bounded if and only if A is relatively compact. Which direction is still always true if X is not complete?

3. Let ℓ^1 denote space of all sequences $(a_n)_n$ of complex numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$, equipped with the metric $d(a, b) = \sum_{n=1}^{\infty} |a_n - b_n|$.
(i) Prove that

$$A = \{a \in \ell^1 \,:\, \sum_{n=1}^{\infty} |a_n| \leq 1\}$$

is bounded and closed, but not compact.
(ii) Let $b \in \ell^1$ with $b_n \geq 0$ for all $n \in \mathbb{N}$. Show that

$$B = \{a \in \ell^1 \,:\, |a_n| \leq b_n \; \forall\, n \in \mathbb{N}\}$$

is compact.

4. For each of the following subsets of $C([0, 1])$ prove or disprove compactness:
(i) $A_1 = \{f \in C([0, 1]) \,:\, \max_{x\in[0,1]} |f(x)| \leq 1\}$,
(ii) $A_2 = \{x \mapsto \sin(\pi n x) \,:\, n \in \mathbb{Z}\}$,
(iii) $A_3 = \{p \,:\, p \text{ polynomial of degree } \leq d\} \cap A_1$ (where $d \in \mathbb{N}$ is given)

5*. Let $C^k([a, b])$ denote the space of k-times continuously differentiable functions on $[a, b]$ endowed with the metric

$$d(f, g) = \sum_{j=0}^{k} \sup_{x\in[a,b]} |f^{(j)}(x) - g^{(j)}(x)|.$$

Let $0 \leq \ell < k$ be integers and consider the canonical embedding map

$$\iota : C^k([a, b]) \to C^\ell([a, b]) \text{ with } \iota(f) = f.$$

Prove that if $B \subset C^k([a, b])$ is bounded, then the image $\iota(B) = \{\iota(f) \,:\, f \in B\} \subset C^\ell([a, b])$ is relatively compact. *Hint:* Use the Arzelà-Ascoli theorem.

Turn the page.

Honors problem 2. For $1 \leq p < \infty$ we denote by ℓ^p the space of sequences $(a_n)_n$ of complex numbers such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Define a metric on ℓ^p by

$$d(a,b) = \left(\sum_{n \in \mathbb{N}} |a_n - b_n|^p \right)^{1/p}.$$

The purpose of this exercise is to prove a theorem of Fréchet that characterizes compactness in ℓ^p. Let $\mathcal{F} \subset \ell^p$.
(i) Assume that $\mathcal{F}$ is bounded and *equisummable* in the following sense: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |a_n|^p < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Then show that $\mathcal{F}$ is totally bounded.
(ii) Conversely, assume that $\mathcal{F}$ is totally bounded. Then show that it is equisummable in the above sense.
Hint: Mimick the proof of Arzelà-Ascoli.