Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 6. Due Monday, March 19.

(Problems with an asterisk (*) are optional.)

1. Let V, W be normed vector spaces and $F : V \to W$ a map.
(i) Show that F is continuous if it is Fréchet differentiable.
(ii) Prove that F is Fréchet differentiable if it is linear and bounded.

2. Let $X = C([0,1])$ be the Banach space of continuous functions on $[0,1]$ (with the supremum norm) and define a map $F : X \to X$ by

$$F(f)(s) = \int_0^s \cos(f(t)^2)dt, \quad s \in [0,1].$$

(i) Show that F is Fréchet differentiable and compute the Fréchet derivative $DF\,|_f$ for each $f \in X$.
(ii) Show that $FX = \{F(f) \,:\, f \in X\} \subset X$ is relatively compact.

3. Let $\mathbb{R}^{n\times n}$ denote the space of real $n \times n$ matrices equipped with the matrix norm $\|A\| = \sup_{\|x\|_2=1} \|Ax\|_2$. Define

$$F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n},\ A \longmapsto A^2.$$

Show that F is totally differentiable and compute $DF\,|_A$.

4. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2}$$

for $(x,y) \neq (0,0)$.
(i) Show that the partial derivatives $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ exist at every point $(x,y) \in \mathbb{R}^2$.
(ii) Show that f is not continuous at $(0,0)$.
(iii) Determine at which points f is totally differentiable.

5*. Let $x \in \mathbb{R}^n$. Recall that $\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}$ for $0 < p < \infty$ and $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$.
(i) Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.
(ii) Show that the limit $\lim_{p\to 0} \|x\|_p$ exists and determine its value.