Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 7. Due Monday, April 2.

(Problems with an asterisk (*) are optional.)

1. Let V be a normed vector space, $U \subseteq V$ open and $F, G : U \to \mathbb{R}$ differentiable maps. Show that the function $F \cdot G : U \to \mathbb{R}$, $(F \cdot G)(x) = F(x)G(x)$ is also differentiable and that

$$D(F \cdot G)\,|_x = F(x) \cdot DG\,|_x + G(x) \cdot DF\,|_x$$

holds for all $x \in U$.

2. Let $U \subseteq \mathbb{R}^n$ be open and convex and $f : U \to \mathbb{R}$ differentiable such that $\partial_1 f(x) = 0$ for all $x \in U$.
(i) Show that the value of $f(x)$ for $x = (x_1, \ldots, x_n) \in U$ does not depend on x_1.
(ii) Does (i) still hold if we assume that U is connected instead of convex? Give a proof or counterexample.

3. Let $A \in \mathbb{R}^{n \times n}$ be a real $n \times n$ matrix such that

$$\|I - A\| < 1,$$

where I is the identity matrix and $\|A\| = \sup_{\substack{x \in \mathbb{R}^n, \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$ the matrix norm. Show that A is invertible. *Hint:* Use the Banach fixed point theorem.

4. Show that there exists a unique $(x, y) \in \mathbb{R}^2$ such that $\cos(\sin(x)) = y$ and $\sin(\cos(y)) = x$.

5. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $\alpha \in \mathbb{R}$ if $f(\lambda x) = \lambda^\alpha f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Suppose that f is differentiable. Then show that f is homogeneous of degree α if and only if

$$\sum_{i=1}^{n} x_i \partial_i f(x) = \alpha f(x)$$

for all $x \in \mathbb{R}^n$. *Hint:* Consider the function $g(\lambda) = f(\lambda x) - \lambda^\alpha f(x)$.

Turn the page.

6*. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function such that $f(x+1, y) = f(x, y+1) = f(x, y)$ for all $x, y \in \mathbb{R}$. Show that for all $v \in \mathbb{R}^2$,

$$\int_0^1 \int_0^1 \partial_v f(x, y) dx dy = 0.$$

(Here ∂_v denotes the directional derivative along v.)

Honors problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $L > 0$ such that $f(x + L) = f(x)$ for all $x \in \mathbb{R}$. Show that for all $x \in \mathbb{R}$ and all $\alpha > 0$ the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x + \alpha t) dt$$

exists and determine its value.
Hint: The limit does not depend on x and α.