Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 8. Due Monday, April 16.

(Problems with an asterisk (*) are optional.)

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that $\partial_x\partial_y f$ and $\partial_y\partial_x f$ exist at every point in $\mathbb{R}^2$, but that $\partial_x\partial_y f(0,0) \neq \partial_y\partial_x f(0,0)$.

2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions (that is, all partial derivatives exist to arbitrary orders). Show that for all multiindices $\alpha \in \mathbb{N}_0^n$ we have

$$\partial^\alpha (f \cdot g)(x) = \sum_{\beta \in \mathbb{N}_0^n \,:\, \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha-\beta} g(x)$$

for all $x \in \mathbb{R}^n$, where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \frac{\alpha_1!\cdots\alpha_n!}{\beta_1!\cdots\beta_n!(\alpha_1-\beta_1)!\cdots(\alpha_n-\beta_n)!}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\partial_1\partial_2 f$ exists everywhere. Does it follow that $\partial_1 f$ exists? Give a proof or counterexample.

4. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = (x^4 - y^4, e^{xy} - e^{-xy}).$$

(i) Compute the Jacobian of F.
(ii) Let $p_0 \in \mathbb{R}^2$ and $p_0 \neq (0,0)$. Show that there exist open neighborhoods $U, V \subset \mathbb{R}^2$ of p_0 and $F(p_0)$, respectively and a function $G : V \to U$ such that $G(F(p)) = p$ for all $p \in U$ and $F(G(p)) = p$ for all $p \in V$.
(iii) Compute $DG\big|_{F(p_0)}$.
(iv) Is F a bijective map?

5. Let $a \in \mathbb{R}$, $a \neq 0$ and $E = \{(x,y,z) \in \mathbb{R}^3 \,:\, a + x + y + z \neq 0\}$ and $f : E \to \mathbb{R}^3$ defined by

$$f(x,y,z) = \left(\frac{x}{a+x+y+z}, \frac{y}{a+x+y+z}, \frac{z}{a+x+y+z}\right).$$

(i) Compute the Jacobian determinant of f (that is, the determinant of the Jacobian matrix).
(ii) Show that f is one-to-one and compute its inverse f^{-1}.