Math 522, Spring 2018 – Analysis II (Roos)
Homework assignment 9. Due Monday, April 23.

(Problems with an asterisk (*) are optional.)

1. Let $f \in C^2(\mathbb{R}^n)$ and suppose that the Hessian of f is positive definite at every point. Show that $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is an injective map.

2. Let $f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle + c$ with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n, c \in \mathbb{R}$. Assume that A is symmetric and positive definite. Show that f has a unique global minimum at some point x_* and determine $f(x_*)$ in terms of A, b, c.

3. Prove that the point x_* from Problem 2 can be computed using gradient descent: that is, if $x_0 \in \mathbb{R}^n$ arbitrary and

$$x_{n+1} = x_n - \alpha \nabla f(x_n)$$

for $n = 0, 1, 2, \ldots$, then the sequence $(x_n)_n$ converges to x_* for all starting points $x_0 \in \mathbb{R}^n$, provided that α is chosen sufficiently small.

4. Look at each of the following as an equation to be solved for $x \in \mathbb{R}$ in terms of parameter $y, z \in \mathbb{R}$. Notice that $(x, y, z) = (0, 0, 0)$ is a solution for each of these equations. For each one, prove that it can be solved for x as a C^1-function of y, z in a neighborhood of $(0, 0, 0)$.

(a) $\cos(x)^2 - e^{\sin(xy)^3+x} = z^2$
(b) $(x^2 + y^3 + z^4)^2 = \sin(x - y + z)$
(c) $x^7 + ye^z x^3 - x^2 + x = \log(1 + y^2 + z^2)$

5*. Suppose that X and Y are *uniformly homeomorphic* metric spaces, i.e. that there exists a bijection $\phi : X \to Y$ such that ϕ and ϕ^{-1} are uniformly continuous.

(a) Prove that X is complete if and only if Y is complete.
(b) Is this still true if we replace "uniformly continuous" by "continuous" in the above definition (i.e. if X and Y are only homeomorphic)? Proof or counterexample.

Turn the page.

Honors problem 4. (Stone-Weierstrass meets neural networks). In the following let $\sigma(t) = e^t$ for $t \in \mathbb{R}$. Fix $n \in \mathbb{N}$ and let $K \subset \mathbb{R}^n$ be a compact set. As usual, let $C(K)$ denote the space of real-valued continuous functions on K. Define a class of functions $\mathcal{N} \subset C(K)$ by saying that $\mu \in \mathcal{N}$ iff there exist $m \in \mathbb{N}$, $W \in \mathbb{R}^{m\times n}$, $v, b \in \mathbb{R}^m$ such that

$$\mu(x) = \sum_{i=1}^{m} \sigma((Wx)_i + b_i)v_i \text{ for all } x \in K.$$

Prove that $\mathcal{N}$ is dense in $C(K)$. [1] *Hint:* Use the Stone-Weierstrass theorem.

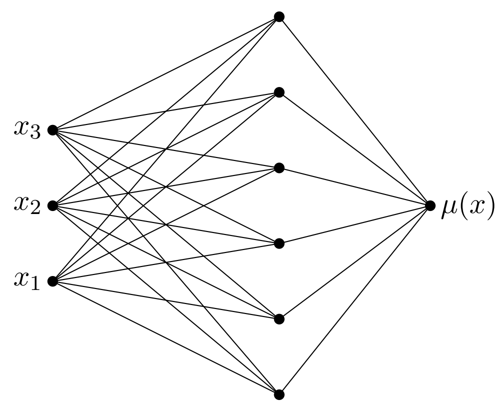

FIGURE 1. Visualization of μ when $n = 3$ and $m = 6$.

[1]As a real-world motivation for this problem, note that a function $\mu \in \mathcal{N}$ can be interpreted as a neural network with a single hidden layer (just in case you happen to know what that is), see Figure 1. Consequently, in this problem you are asked to show that every continuous function can be uniformly approximated by neural networks of this form. Results of this sort are important theoretical foundations for machine learning.