Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 1. Due Monday, September 17.

1. Prove or disprove convergence for each of the following sequences and in case of convergence, determine the limit:
(i) $a_n = \sqrt{n^4 + \cos(n^2)} - n^2$
(ii) $b_n = n^2 + \frac{1}{2}n - \sqrt{n^4 + n^3}$
(iii) $c_n = \sum_{k=n}^{n^2} \frac{1}{k}$
(iv) $d_n = n \sum_{k=0}^{\infty} \frac{1}{n^2+k^2}$

2. Find differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n = 1, 2, 3, \ldots$ such that $(f_n)_n$ converges uniformly on $\mathbb{R}$, but $(f_n'(x))_n$ does not converge for any $x \in \mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function (i.e. derivatives of all orders exist). Assume that there exist $A > 0, R > 0$ such that

$$|f^{(n)}(x)| \le A^n n!$$

for $|x| < R$. Show that there exists $r > 0$ such that for every $|x| < r$ we have that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

(That is, prove that the series on the right hand side converges and that the limit is $f(x)$.)

4. For a positive real number x define

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n(n+1) + x}.$$

(i) Show that $f : (0, \infty) \to (0, \infty)$ is a well-defined and continuous function.
(ii) Prove that there exists a unique $x_0 \in (0, \infty)$ such that $f(x_0) = 2\pi$.
(iii) Determine the value of x_0.

5. For which $x \in \mathbb{R}$ do the following series converge? On which sets do these series converge uniformly?

$$\text{(i)} \sum_{n=1}^{\infty} n^{10} x^n, \qquad \text{(ii)} \sum_{n=1}^{\infty} (2^{1/n} - 1)^n x^n \qquad \text{(iii)} \sum_{n=1}^{\infty} \tan(n^{-2}) e^{nx}.$$