Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 10. Due Monday, December 3.

(Problems with an asterisk (*) are optional.)

1. We define the subset $A \subset \mathbb{R}$ as follows: $x \in A$ if and only if there exists $c > 0$ such that
$$|x - j2^{-k}| \geq c2^{-k}$$
holds for all $j \in \mathbb{Z}$ and integers $k \geq 0$. Show that A is meager and dense.

2. (i) Show that if X is a normed vector space and $U \subset X$ a proper subspace, then U has empty interior.
(ii) Let
$$X = \{P : \mathbb{R} \to \mathbb{R} \,|\, P \text{ is a polynomial}\}.$$
Use the Baire category theorem to prove that there exists no norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is a Banach space.

3. Consider $X = C([-1, 1])$ with the usual norm $\|f\|_\infty = \sup_{t\in[-1,1]} |f(t)|$. Let
$$A_+ = \{f \in X \,:\, f(t) = f(-t) \quad \forall t \in [-1, 1]\},$$
$$A_- = \{f \in X \,:\, f(t) = -f(-t) \quad \forall t \in [-1, 1]\}.$$
(i) Show that A_+ and A_- are meager.
(ii) Is $A_+ + A_- = \{f + g \,:\, f \in A_+, g \in A_-\}$ meager?

4. Fix $L > 0$. Show that the set
$$A = \{f \in C([0, 1]) \,:\, \exists\, t \in [0, 1] \text{ s.t. } |f(t) - f(s)| \leq L|t - s| \;\forall\, s \in [0, 1]\}$$
has empty interior in $C([0, 1])$.
Hint: Show that the complement of A is dense by approximating an arbitrary continuous function by "sawtooth" functions with arbitrarily large slope.

Turn the page.

Definition. A set $A \subset \mathbb{R}$ is called a *Lebesgue null set* if for every $\varepsilon > 0$ there exist intervals $I_1, I_2, \ldots$ such that

$$A \subset \bigcup_{j=1}^{\infty} I_j \text{ and } \sum_{j=1}^{\infty} |I_j| \leq \varepsilon.$$

(Here $|I|$ denotes the length of the interval I.)

5*. (i) Show that every countable subset of $\mathbb{R}$ is a Lebesgue null set.
(ii) Show that

$$C = [0,1] \setminus \bigcup_{\ell=0}^{\infty} \bigcup_{k=0}^{3^\ell - 1} \left(\tfrac{3k+1}{3^{\ell+1}}, \tfrac{3k+2}{3^{\ell+1}}\right)$$

is a Lebesgue null set.
(iii) Show that the set A from Problem 1 above is a Lebesgue null set.

6*. Give an example of a comeager Lebesgue null set. (Recall that a set is called *comeager* if its complement is meager.)