Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 3. Due Monday, October 1.

(Problems with an asterisk (*) are optional.)

1. Let $(f_n)_n$ be a sequence of continuous functions on $[0,1]$ and f a continuous function on $[0,1]$. Assume that $\|f_n - f\|_2 \to 0$. Does it follow that $f_n(x) \to f(x)$ for some $x \in [0,1]$? Give a proof or counterexample.

2. (i) Let $(a_k)_k$ be a sequence of real or complex numbers with limit L. Prove that

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = L$$

Definition: Given the sequence a_k, form the partial sums $s_n = \sum_{k=1}^{n} a_k$ and let

$$\sigma_N = \frac{s_1 + \cdots + s_N}{N}.$$

σ_N is called the Nth Cesàro mean of the sequence s_k or the Nth Cesàro sum of the series $\sum_{k=1}^{\infty} a_k$. If σ_N converges to a limit S we say that the series $\sum_{k=1}^{\infty} a_k$ is Cesàro summable to S.

(ii) Prove that if $\sum_{k=1}^{\infty} a_k$ is summable to S (i.e. by definition converges with sum S) then $\sum_{k=1}^{\infty} a_k$ is Cesàro summable to S.

(iii) Prove that the sum $\sum_{k=1}^{\infty} (-1)^{k-1}$ does not converge but is Cesàro summable to some limit S and determine S.

3. Show that there exists no continuous 1-periodic function g such that $f * g = f$ holds for all continuous 1-periodic functions f.
Hint: Use the Riemann-Lebesgue lemma.

4. Define a sequence of polynomials $(T_n)_n$ by $T_0(x) = 1$, $T_1(x) = x$ and the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.
(i) Show that $T_n(x) = \cos(nt)$ if $x = \cos(t)$.
Hint: Use that $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$ for all $a, b \in \mathbb{C}$.
(ii) Compute

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}}$$

for all non-negative integers n, m.
(iii) Prove that $|T_n(x)| \leq 1$ for $x \in [-1,1]$ and determine when there is equality.

5*. Construct a sequence of real-valued polynomials $(p_n)_n$ such that

$$\int_0^1 p_n(x) p_m(x)\,dx = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m \end{cases}$$

for all non-negative integers n, m.

Turn the page.

Honors problem 2. The purpose of this exercise is to play with a construction of L^2 that does not rely on the Lebesgue integral. Let (X, d) be a metric space. Recall that the *completion* $\overline{X}$ of X is defined as follows: for two Cauchy sequences $(a_n)_n$, $(b_n)_n$ in X we say that $(a_n)_n \sim (b_n)_n$ if $\lim_{n\to\infty} d(a_n, b_n) = 0$. Then $\sim$ is an equivalence relation on the space of Cauchy sequences and we define $\overline{X}$ as the set of equivalence classes. As usual, we identify X with a subset of $\overline{X}$ by identifying $x \in X$ with the equivalence class of the constant sequence $x, x, \ldots$. We make $\overline{X}$ a metric space by defining

$$d(a, b) = \lim_{n\to\infty} d(a_n, b_n),$$

where $(a_n)_n, (b_n)_n$ are representatives of $a, b \in \overline{X}$, respectively. Then $\overline{X}$ is a complete metric space. Let us denote by $L^2_{\mathrm{c}}(a, b)$ the metric space of continuous functions on $[a, b]$ equipped with the metric $d(f, g) = \|f - g\|_2$, where $\|f\|_2 = (\int_a^b |f|^2)^{1/2}$. Define

$$L^2(a, b) = \overline{L^2_c(a, b)}.$$

(i) Define an inner product on $L^2(a, b)$ by

$$\langle f, g\rangle = \lim_{n\to\infty} \int_a^b f_n(x)\overline{g_n(x)}dx,$$

for $f, g \in L^2(a, b)$ with $(f_n)_n, (g_n)_n$ being representatives of f, g, respectively. Show that this is well-defined: that is, show that the limit on the right hand side exists and is independent of the representatives $(f_n)_n, (g_n)_n$ and that $\langle\cdot,\cdot\rangle$ is an inner product.

Hint: Use the Cauchy-Schwarz inequality on $L^2_c(a, b)$.

For $f \in L^2(a, b)$ we define $\|f\|_2 = \langle f, f\rangle^{1/2}$. Let $(\phi_n)_{n=1,2,\ldots}$ be an orthonormal system in $L^2(a, b)$ (that is, $\langle\phi_n, \phi_m\rangle = 0$ if $n \neq m$ and $= 1$ if $n = m$).

(ii) Prove Bessel's inequality: for every $f \in L^2(a, b)$ we have

$$\sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 \leq \|f\|_2^2$$

Hint: Use the same proof as seen for $L^2_c(a, b)$ in the lecture!

(iii) Let $(c_n)_n \subset \mathbb{C}$ be a sequence of complex numbers and let

$$f_N = \sum_{n=1}^{N} c_n\phi_n \in L^2(a, b).$$

Show that $(f_N)_N$ converges in $L^2(a, b)$ if and only if

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty.$$