Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 4. Due Monday, October 15.

(Problems with an asterisk (*) are optional.)

1. Let f be the 1-periodic function such that $f(x) = |x|$ for $x \in [-1/2, 1/2]$. Determine explicitly a sequence of trigonometric polynomials $(p_N)_N$ such that $p_N \to f$ uniformly as $N \to \infty$.

2. Let $f \in C([0,1])$ and $\mathcal{A} \subset C([0,1])$ dense. Suppose that

$$\int_0^1 f(x)\overline{a(x)}dx = 0$$

for all $a \in \mathcal{A}$. Show that $f = 0$. *Hint:* Show that $\int_0^1 |f(x)|^2 dx = 0$.

3. Prove that

$$-\frac{1}{2} = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(n)}{n}.$$

4*. Let f, g be continuous, 1-periodic functions. Recall that the Fourier coefficents of f are defined by $\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi int}dt$.
(i) Show that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$.
(ii) Show that $\widehat{f \cdot g}(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(n-m)\widehat{g}(m)$.
(iii) If f is continuously differentiable, prove that $\widehat{f'}(n) = 2\pi in\widehat{f}(n)$.
(iv) Let $y \in \mathbb{R}$ and set $f_y(x) = f(x+y)$. Show that $\widehat{f_y}(n) = e^{2\pi iny}\widehat{f}(n)$.
(v) Let $m \in \mathbb{Z}$, $m \neq 0$ and set $f_m(x) = f(mx)$. Show that $\widehat{f_m}(n)$ equals $\widehat{f}(\frac{n}{m})$ if m divides n and zero otherwise.

5. Let f be 1-periodic and k times continuously differentiable. Prove that there exists a constant $c > 0$ such that

$$|\widehat{f}(n)| \leq c|n|^{-k} \quad \text{for all } n \in \mathbb{Z}.$$

Hint: What can you say about the Fourier coefficients of $f^{(k)}$?

Turn the page.

6. Let f be 1-periodic and continuous.
(i) Suppose that $\widehat{f}(n) = -\widehat{f}(-n) \geq 0$ holds for all non-negative integers n. Prove that

$$\sum_{n=1}^{\infty} \frac{\widehat{f}(n)}{n} < \infty.$$

(ii) Show that there does not exist a 1-periodic continuous function f such that

$$\widehat{f}(n) = \frac{\operatorname{sgn}(n)}{\log |n|} \quad \text{for all } n \geq 2.$$

Here $\operatorname{sgn}(n) = 1$ if $n > 0$ and $\operatorname{sgn}(n) = -1$ if $n < 0$.

7*. Recall Fejér's theorem: continuous 1-periodic functions can be uniformly approximated by trigonometric polynomials. Using Fejér's theorem and Taylor expansion for trigonometric functions, give another proof of Weierstrass' theorem: every continuous function on $[0, 1]$ can be uniformly approximated by polynomials.