Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 5. Due Monday, October 22.

(Problems with an asterisk (*) are optional.)

1. Let K be a finite set and $\mathcal{A}$ a family of functions on K that is a self-adjoint algebra, separates points and vanishes nowhere. Show that $\mathcal{A}$ must then already contain every function on K.

2. Let $x \in \mathbb{R}^n$. Define $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $0 < p < \infty$ and $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$.
(i) Show that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.
(ii) Show that the limit $\lim_{p\to 0} \|x\|_p$ exists and determine its value.

3. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on a normed vector space X.
(i) Show that a sequence $(x_n)_n \subset X$ converges with respect to $\|\cdot\|_1$ if and only if it converges with respect to $\|\cdot\|_2$.
(ii) Show that a set $U \subset X$ is open with respect to $\|\cdot\|_1$ if and only if it is open with respect to $\|\cdot\|_2$.

4. Let $C(\mathbb{R})$ be the set of continuous functions on $\mathbb{R}$. Let $w(t) = \frac{t}{1+t}$ for $t \geq 0$. Define

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} w\Big(\sup_{x\in[-k,k]} |f(x) - g(x)|\Big).$$

(i) Show that d is a well-defined metric.
(ii) Show that $C(\mathbb{R})$ is complete with this metric.
(iii) Show that there exists no norm $\|\cdot\|$ on $C(\mathbb{R})$ such that $d(f,g) = \|f-g\|$.

Turn the page.

Honors Problem 3. Let $\sigma(t) = e^t$ for $t \in \mathbb{R}$. Fix $n \in \mathbb{N}$ and let $K \subset \mathbb{R}^n$ be a compact set. As usual, let $C(K)$ denote the space of real-valued continuous functions on K. Define a class of functions $\mathcal{N} \subset C(K)$ by saying that $\mu \in \mathcal{N}$ iff there exist $m \in \mathbb{N}$, $W \in \mathbb{R}^{m \times n}$, $v, b \in \mathbb{R}^m$ such that

$$\mu(x) = \sum_{i=1}^{m} \sigma((Wx)_i + b_i)v_i \text{ for all } x \in K.$$

Prove that $\mathcal{N}$ is dense in $C(K)$. [1]

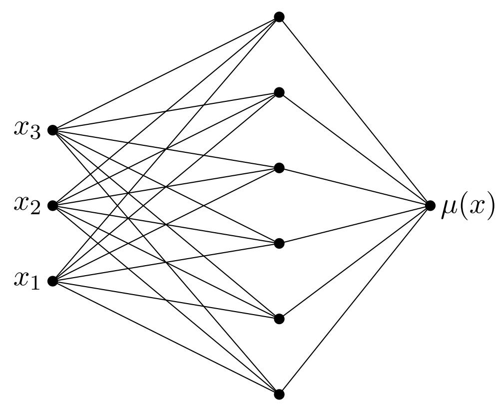

FIGURE 1. Visualization of μ when $n = 3$ and $m = 6$.

[1]As a real-world motivation for this problem, note that a function $\mu \in \mathcal{N}$ can be interpreted as a neural network with a single hidden layer, see Figure 1. Consequently, in this problem you are asked to show that every continuous function can be uniformly approximated by neural networks of this form.