Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 6. Due Monday, October 29.

(Problems with an asterisk (*) are optional.)

1. Let $X = C([0,1])$ be the Banach space of continuous functions on $[0,1]$ (with the supremum norm) and define a map $F : X \to X$ by

$$F(f)(s) = \int_0^s \cos(f(t)^2)dt, \quad s \in [0,1].$$

(i) Show that F is Fréchet differentiable and compute the Fréchet derivative $DF\,|_f$ for each $f \in X$.
(ii) Show that $FX = \{F(f) \,:\, f \in X\} \subset X$ is relatively compact.

2. Let $\mathbb{R}^{n\times n}$ denote the space of real $n \times n$ matrices equipped with the matrix norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$. Define

$$F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n},\ A \longmapsto A^2.$$

Show that F is totally differentiable and compute $DF\,|_A$.

3. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x) = \frac{x_1 x_2}{x_1^2 + x_2^2}$ if $x \neq 0$ and $F(0) = 0$.
(i) Show that the partial derivatives $\partial_1 F(x)$, $\partial_2 F(x)$ exist for every $x \in \mathbb{R}^2$.
(ii) Show that F is not continuous at $(0,0)$.
(iii) Determine at which points F is totally differentiable.

4. Let X be a normed vector space, $U \subset X$ open and $F, G : U \to \mathbb{R}$ differentiable at $x \in U$. Show that the function $F \cdot G : U \to \mathbb{R}$, $(F \cdot G)(x) = F(x)G(x)$ is also differentiable at x and that

$$D(F \cdot G)\,|_x = F(x) \cdot DG\,|_x + G(x) \cdot DF\,|_x\,.$$

5*. Let X be the space of continuous functions on $[0,1]$ equipped with the norm $\|f\| = \int_0^1 |f(t)|dt$. Define a linear map $T : X \to X$ by

$$Tf(x) = \int_0^x f(t)dt.$$

Show that T is well-defined and bounded and determine the value of $\|T\|_{\mathrm{op}}$.