Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 7. Due Monday, November 5.

(Problems with an asterisk (*) are optional.)

1. Show that there exists a unique $(x, y) \in \mathbb{R}^2$ such that $\cos(\sin(x)) = y$ and $\sin(\cos(y)) = x$.

2. Let $U \subseteq \mathbb{R}^n$ be open and convex and $f : U \to \mathbb{R}$ differentiable such that $\partial_1 f(x) = 0$ for all $x \in U$.
(i) Show that the value of $f(x)$ for $x = (x_1, \dots, x_n) \in U$ does not depend on x_1.
(ii) Does (i) still hold if we assume that U is connected instead of convex? Give a proof or counterexample.

3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $\alpha \in \mathbb{R}$ if $f(\lambda x) = \lambda^\alpha f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Suppose that f is differentiable. Then show that f is homogeneous of degree α if and only if

$$\sum_{i=1}^n x_i \partial_i f(x) = \alpha f(x)$$

for all $x \in \mathbb{R}^n$. *Hint:* Consider the function $g(\lambda) = f(\lambda x) - \lambda^\alpha f(x)$.

4. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = (x^4 - y^4, e^{xy} - e^{-xy}).$$

(i) Compute the Jacobian of F.
(ii) Let $p_0 \in \mathbb{R}^2$ and $p_0 \neq (0, 0)$. Show that there exist open neighborhoods $U, V \subset \mathbb{R}^2$ of p_0 and $F(p_0)$, respectively and a function $G : V \to U$ such that $G(F(p)) = p$ for all $p \in U$ and $F(G(p)) = p$ for all $p \in V$.
(iii) Compute $DG\big|_{F(p_0)}$.
(iv) Is F a bijective map?

5. Let $a \in \mathbb{R}$, $a \neq 0$ and $E = \{(x, y, z) \in \mathbb{R}^3 : a + x + y + z \neq 0\}$ and $f : E \to \mathbb{R}^3$ defined by

$$f(x, y, z) = \left(\frac{x}{a + x + y + z}, \frac{y}{a + x + y + z}, \frac{z}{a + x + y + z} \right).$$

(i) Compute the Jacobian determinant of f (that is, the determinant of the Jacobian matrix).
(ii) Show that f is one-to-one and compute its inverse f^{-1}.

Turn the page.

6*. Prove that there exists $\delta > 0$ such that for all square matrices $A \in \mathbb{R}^{n\times n}$ with $\|A - I\| < \delta$ (where I denotes the identity matrix) there exists $B \in \mathbb{R}^{n\times n}$ such that $B^2 = A$.

Honors problem 4. Let $L \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ (i.e. all partial derivatives of L up to order 2 exist on $(a,b) \times \mathbb{R} \times \mathbb{R}$ and are continuous on $[a,b] \times \mathbb{R} \times \mathbb{R}$). For $f : [a,b] \to \mathbb{R}$ we define

$$\mathscr{I}(f) = \int_a^b L(t, f(t), f'(t))dt.$$

Let $\mathcal{A} = \{f \in C^2([a,b]) \,:\, f(a) = f(b) = 0\}$ and assume that $f_* \in \mathcal{A}$ is such that

$$\mathscr{I}(f_*) = \inf_{f\in\mathcal{A}} \mathscr{I}(f).$$

Prove that

$$\partial_2 L(t, f_*(t), f_*'(t)) - \frac{d}{dt}\partial_3 L(t, f_*(t), f_*'(t)) = 0.$$

Hint: Consider perturbations $g_\varepsilon(t) = f_*(t) + \varepsilon h(t)$ and differentiate in ε.