Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 8. Due Monday, November 19.

(Problems with an asterisk (*) are optional.)

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that $\partial_x \partial_y f$ and $\partial_y \partial_x f$ exist at every point in $\mathbb{R}^2$, but that $\partial_x \partial_y f(0,0) \neq \partial_y \partial_x f(0,0)$.

2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions (that is, all partial derivatives exist to arbitrary orders and are continuous). Show that for all multiindices $\alpha \in \mathbb{N}_0^n$ we have

$$\partial^\alpha (f \cdot g)(x) = \sum_{\beta \in \mathbb{N}_0^n \,:\, \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha - \beta} g(x)$$

for all $x \in \mathbb{R}^n$, where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \frac{\alpha_1! \cdots \alpha_n!}{\beta_1! \cdots \beta_n! (\alpha_1 - \beta_1)! \cdots (\alpha_n - \beta_n)!}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\partial_1 \partial_2 f$ exists everywhere. Does it follow that $\partial_1 f$ exists? Give a proof or counterexample.

4. Consider the initial value problem

$$\begin{cases} y'(t) = e^{y(t)^2} - \frac{1}{ty(t)}, \\ y(1) = 1. \end{cases}$$

Find an interval $I = (1-h, 1+h)$ such that this problem has a unique solution y in I. Give an explicit estimate for h (it does not need to be best possible).

5. Consider the initial value problem

$$\begin{cases} y'(t) = t + \sin(y(t)), \\ y(2) = 1. \end{cases}$$

Find the largest interval $I \subseteq \mathbb{R}$ containing $t_0 = 2$ such that the problem has a unique solutions y in I.

Turn the page.

6*. For a function $f : [a, b] \to \mathbb{R}$ define

$$\mathscr{I}(f) = \int_a^b (1 + f'(t)^2)^{1/2} dt.$$

Let $\mathcal{A} = \{f \in C^2([a, b]) \,:\, f(a) = c, f(b) = d\}$. Determine $f_* \in \mathcal{A}$ such that

$$\mathscr{I}(f_*) = \inf_{f \in \mathcal{A}} \mathscr{I}(f).$$

What is the geometric meaning of $\mathscr{I}(f)$ and $\inf_{f \in \mathcal{A}} \mathscr{I}(f)$?