Math 522, Fall 2018 – Analysis II (Roos)
Homework assignment 9. Due Monday, November 26.

(Problems with an asterisk (*) are optional.)

1. Let $f \in C^2(\mathbb{R}^n)$ and suppose that the Hessian of f is positive definite at every point. Show that $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is an injective map.

2. Let $\mathcal{D} \subset \mathbb{R}^2$ be a finite set. Define a function $E : \mathbb{R}^3 \to \mathbb{R}$ by

$$E(a,b,c) = \sum_{x \in \mathcal{D}} (ax_1^2 + bx_1 + c - x_2)^2.$$

(a) Show that E is convex.
(b) Does there exist a set $\mathcal{D}$ such that E is strongly convex? Proof or counterexample.

3. Let $f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle + c$ with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n, c \in \mathbb{R}$. Assume that A is symmetric and positive definite. Show that f has a unique global minimum at some point x_* and determine $f(x_*)$ in terms of A, b, c.

4. (a) Find a convex function that is not bounded from below.
(b) Find a strictly convex function that is not bounded from below.
(c) If a function is strictly convex and bounded from below, does it necessarily have a critical point? (Proof or counterexample.)

5. Let $f \in C^2(\mathbb{R}^n)$. Recall that we defined f to be *strongly convex* if there exists $\beta > 0$ such that $\langle D^2 f|_x y, y\rangle \geq \beta \|y\|^2$ for every $x, y \in \mathbb{R}^n$. Show that f is strongly convex if and only if there exists $\gamma > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \gamma t(1-t)\|x-y\|^2$$

for all $x, y \in \mathbb{R}^n, t \in [0,1]$.
(Consequently, that condition can serve as an alternative definition of strong convexity, which is also valid if f is not C^2.)

6*. (a) Give an example of a convex function that is not continuous.
(b) Let $f : (a,b) \to \mathbb{R}$. Show that if f is convex, then f is continuous.

7*. Construct a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at x for every $x \in \mathbb{Q}$.