Math 629, Spring 2019 – Homework 10.
Due Monday, April 15.

(Problems with an asterisk (*) are optional.)

1. Do Exercise 14 in Chapter 3.5 in Stein-Shakarchi "Real Analysis".

2. For $a, b \in \mathbb{R}$ define the function $F_{a,b} : [0,1] \to \mathbb{R}$ by

$$F_{a,b}(x) = \begin{cases} x^a \sin(x^b) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine all values of a, b such that the function $F_{a,b}$ is of bounded variation on $[0,1]$.

3. Let $F : [a,b] \to \mathbb{R}$ be a continuous function of bounded variation. Show that the total variation on $[a,x]$ defined by

$$T_F(a,x) = \sup_{a=t_0<\cdots<t_N=x} \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|$$

is a continuous function of $x \in [a,b]$.

4. Suppose that f is an integrable function on $\mathbb{R}$. Let

$$Mf(x) = \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x-t)| dt$$

denote the centered Hardy-Littlewood maximal function.
(i) Show that Mf is lower semicontinuous.
Recall: A real-valued function g is lower semicontinuous if and only if the superlevel set $\{g > \lambda\}$ is open for all $\lambda \in \mathbb{R}$.
(ii*) Suppose f is continuous at $x \in \mathbb{R}$. Prove that Mf is continuous at x.

5*. (This exercise is meant for those that know some complex analysis.) Let $a > 1$. The function $t \mapsto e^{it^a}\mathbf{1}_{t>0}$ is not integrable. Nevertheless, show that

$$\int_0^\infty e^{it^a} dt := \lim_{R\to\infty} \int_0^R e^{it^a} dt = e^{i\frac{\pi}{2a}} \Gamma\big(1 + \frac{1}{a}\big),$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ denotes the *gamma function.*
Hint: Apply Cauchy's theorem along a conveniently chosen contour in the complex plane to compare with $\int_0^\infty e^{-t^a} dt$.