Math 629, Spring 2019 – Homework 11.
Due Monday, April 22.

(Problems with an asterisk (*) are optional.)

1. Let $f : (a, b) \times \mathbb{R}^d \to \mathbb{R}$ be such that

- for every $y \in \mathbb{R}^d$, $x \mapsto f(x, y)$ is differentiable,
- for every $x \in (a, b)$, $y \mapsto f(x, y)$ is integrable, and
- there exists a non-negative integrable function $g : \mathbb{R}^d \to \mathbb{R}$ such that $|\partial_x f(x, y)| \le g(y)$ for every $x \in (a, b), y \in \mathbb{R}^d$.

For $x \in (a, b)$ let

$$F(x) = \int f(x, y)dy.$$

Prove that $F : (a, b) \to \mathbb{R}$ is differentiable and that for every $x \in (a, b)$,

$$F'(x) = \int \partial_x f(x, y)dy.$$

2. Let $\varphi : [a, b] \to \mathbb{R}$ be absolutely continuous and strictly increasing. Prove that for every integrable function f we have

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b f(\varphi(x))\varphi'(x)dx$$

3. Suppose that F and G are absolutely continuous on $[a, b]$.

(i) Prove that $F \cdot G$ is also absolutely continuous.

(ii) Prove that

$$\int_a^b F' \cdot G + \int_a^b F \cdot G' = F(b)G(b) - F(a)G(a).$$

4. Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous.

(i) Suppose $E \subset [a, b]$ has measure zero. Show that $m(F(E)) = 0$.

(ii) Show that $F(E)$ is measurable for every measurable set $E \subset [a, b]$.

(Turn the page.)

5*. Suppose F is absolutely continuous on $[a,b]$. Show that

$$\sup_{a=t_0<\cdots<t_N=b} \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})| = \int_a^b |F'|$$

Hint: First prove it for C^1 functions (recognize a Riemann sum) and then use an approximation argument.

6*. Let $F : [a,b] \to \mathbb{R}$ be a function of bounded variation. Given $g : [a,b] \to \mathbb{R}$ we say that g is *Stieltjes integrable* with respect to F if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{t_j : j = 0,\dots,N\}$ and $\{t'_j : j = 0,\dots,N'\}$ are two partitions of $[a,b]$ with $\max_j |t_j - t_{j-1}| \le \delta$, $\max_j |t'_j - t'_{j-1}| \le \delta$, then

$$\Big|\sum_{j=1}^{N} g(\xi_j)(F(t_j) - F(t_{j-1})) - \sum_{j=1}^{N'} g(\xi'_j)(F(t'_j) - F(t'_{j-1}))\Big| \le \varepsilon$$

holds for every choice of $\xi_j \in [t_{j-1}, t_j]$ and $\xi'_j \in [t'_{j-1}, t'_j]$.
(i) Suppose that g is Stieltjes integrable with respect to F as defined above. Prove that there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition $\{t_j : j = 0,\dots,N\}$ of $[a,b]$ with $\max_j(t_j - t_{j-1}) \le \delta$ and every choice of $\xi_j \in [t_{j-1}, t_j]$ we have

$$\Big|\sum_{j=1}^{N} g(\xi_j)(F(t_j) - F(t_{j-1})) - L\Big| \le \varepsilon.$$

From now on we write this number as

$$L = \int_a^b g dF$$

and call it the *Stieltjes integral* of g with respect to F.
(ii) Prove that continuous functions are Stieltjes integrable.
(iii) Suppose that $f \in L^1([a,b])$ (that is, $f\mathbf{1}_{[a,b]} \in L^1(\mathbb{R})$). Let $F(x) = \int_a^x f$. Show that the Lebesgue integral $\int_a^b f$ can be written as a Stieltjes integral (i.e. find F, g such that $\int_a^b f = \int_a^b g dF$).
Hint: (iii) is easier than you may think.