Math 629, Spring 2019 – Homework 3.
Due Monday, February 18.

Do Exercises **37, 22, 23, 36** in Chapter 1.6[1].

Honors problem. The purpose of this exercise is to give a small taste of abstract measure theory (if this catches your interest, read more in Chapter 6 in the book). Let X be a set and $\mathcal{P}(X)$ the set of all subsets of X. Let us call a map $\mu : \mathcal{P}(X) \to [0, \infty]$ an *outer measure* if it satisfies

1. $\mu(\emptyset) = 0$.
2. $E \subset E' \subset X$ implies $\mu(E) \le \mu(E')$
3. If $(E_j)_{j=1,2,\ldots}$ are subsets of X, then

$$\mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) \le \sum_{j=1}^{\infty} \mu(E_j).$$

(a) Let $\mathcal{Q} \subset \mathcal{P}(X)$ be an *arbitrary* collections of subsets of X and let $\mu_0 : \mathcal{Q} \to [0, \infty)$ be an *arbitrary* map. Then, for every $E \subset X$ we define

$$\mu(E) = \inf_{E \subset \bigcup_{j=1}^{\infty} Q_j} \sum_{j=1}^{\infty} \mu_0(Q_j),$$

where the infimum is taken over all coverings of E by countable collections $(Q_j)_{j=1,2,\ldots} \subset \mathcal{Q}$ of sets in $\mathcal{Q}$. Prove that μ is an outer measure.
(b) At this level of generality it is not necessarily true that $\mu(Q) = \mu_0(Q)$ for every $Q \in \mathcal{Q}$. Prove that this holds if and only if $\mathcal{Q}$ and μ_0 are chosen such that

$$\mu_0(Q) \le \sum_{j=1}^{\infty} \mu_0(Q_j)$$

holds for every $Q, Q_1, Q_2, \cdots \in \mathcal{Q}$ with $Q \subset \cup_{j=1}^{\infty} Q_j$.

Optional (not graded): Read about the construction of a non-measurable set and the axiom of choice on p. 24-26 and do Exercises 33 and 21.

[1]Numbering refers to the textbook *Real Analysis* by E. M. Stein and R. Shakarchi.