Math 629, Spring 2019 – Homework 5.
Due Monday, March 4.

(Problems with an asterisk (*) are optional.)

0*. Do as many of the exercises in Chapter 2.5[1] as you can.

1*. Let $g \geq 0$ be an integrable function on $\mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ be an open set. Define a function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$f(x) = \int_{\mathbb{R}^d} g(y) \mathbf{1}_{x+U}(y) dy.$$

(a) Prove that f is lower semi-continuous (that is, $\liminf_{x \to x_0} f(x) \geq f(x_0)$ for all $x \in \mathbb{R}^d$).
(b) Is f necessarily upper semi-continuous (that is, $\limsup_{x \to x_0} f(x) \leq f(x_0)$ for all $x \in \mathbb{R}^d$) ? (Proof or counterexample.)
(c) Does the conclusion in (a) still hold if we only assume that U is measurable, rather than open? (Proof or counterexample.)

2. Construct an integrable function $f : \mathbb{R} \to [0, \infty)$ such that f is unbounded on every open interval.
Hint: What is the easiest unbounded integrable function you can think of? Can you go from there to make an integrable function that blows up whenever approaching a rational number?

3. Let $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Prove that for every $\lambda > 0$ we have

$$m(\{x \in \mathbb{R}^d : |f(x)| > \lambda\})^{1/p} \leq \lambda^{-1} \|f\|_p.$$

4. Let $1 \leq p < \infty$. Let $(f_n)_n$ be a sequence in $L^p(\mathbb{R}^d)$ that converges to some $f \in L^p(\mathbb{R}^d)$ (in L^p-norm).
(a) Prove that for every $\varepsilon > 0$ we have

$$\lim_{n \to \infty} m(\{x \in \mathbb{R}^d : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

(b) Construct a sequence $(f_n)_n$ convergent in $L^p(\mathbb{R}^d)$ such that $\lim_{n \to \infty} f_n(x)$ does not exist for a.e. $x \in \mathbb{R}^d$.

5*. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable[2] function. Prove that f is Lebesgue integrable on $[a, b]$ and that the Riemann integral of f is equal to the Lebesgue integral of f.
Hint: First try this yourself and if necessary, refer to Ch. 2, Thm. 1.5 in the book.

Turn the page.

[1] Numbering refers to the textbook *Real Analysis* by E. M. Stein and R. Shakarchi.

[2] Here you may define Riemann integrals via Riemann sums or upper and lower sums, whichever you prefer.

Honors problem 3. Let X be a set, $\mathcal{Q} \subset \mathcal{P}(X)$, $\mu_0 : \mathcal{Q} \to [0, \infty)$ and $\mu : \mathcal{P}(X) \to [0, \infty]$ the outer measure generated by μ_0 (see Homework 3). We call a set $E \subset X$ *Carathéodory measurable* (with respect to μ) if for every $Q \in \mathcal{Q}$ we have

$$\mu(Q) = \mu(Q \cap E) + \mu(Q \cap E^c).$$

(a) Let Σ_{C} denote the set of Carathéodory measurable sets. Show that Σ_{C} is a σ-algebra.
(One can also show that the restriction $\mu|_{\Sigma_{\mathrm{C}}}$ is a *measure*; i.e. countably additive, see Thm. 1.1 in Ch. 6.)
(b) Let $X = \mathbb{R}^d$ and let $\mathcal{Q}$ denote the collection of closed cubes in $\mathbb{R}^d$ and set $\mu_0(Q) = |Q|$ for all $Q \in \mathcal{Q}$ (where $|Q|$ denotes the volume of the cube Q). Prove that $E \subset \mathbb{R}^d$ is Carathéodory measurable if and only if it is Lebesgue measurable (as defined in the lecture).
(c) Let $X = \mathbb{R} \times (0, \infty)$ denote the upper half-plane and let $\mathcal{Q}$ be the set of all *tents* which we define as sets $T = T(x, s) \subset X$ of the form

$$T(x, s) = \{(y, t) \in X \,:\, t < s,\, |x - y| < s - t\}$$

for some $(x, s) \in X$. Let $\mu_0(T(x, s)) = s$. Prove that $\Sigma_{\mathrm{C}} = \{\emptyset, X\}$ and that $\mu(T) = \mu_0(T)$ for all tents T.