Math 629, Spring 2019 – Homework 6.
Due Monday, March 11.

(Problems with an asterisk (*) are optional.)

1. Show that $L^p(\mathbb{R}^d) \not\subset L^q(\mathbb{R}^d)$ for all $p, q \in [1, \infty]$ with $p \neq q$.

2. (a) Let $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty)$ and assume that f is supported on a set of finite measure. Show that $f \in L^q(\mathbb{R}^d)$ for all $1 \leq q \leq p$.
(b) Let $g \in L^p(\mathbb{R})$ for some $p \in [1, \infty)$ and assume that g is constant a.e. on every interval of the form $[k, k+1]$ for $k \in \mathbb{Z}$. Show that $g \in L^q(\mathbb{R})$ for every $p \leq q \leq \infty$.

3. Let $1 \leq p < r < q \leq \infty$ and assume that $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Prove that

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}$$

where $\theta \in (0, 1)$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. In particular, $f \in L^r(\mathbb{R}^d)$.

4. Let $p \in [1, \infty)$ and $(f_n)_{n \in \mathbb{N}} \subset L^p(\mathbb{R}^d)$ such that $\|f_n\|_p \leq n^{-2}$ for all $n \in \mathbb{N}$. Does $(f_n)_{n \in \mathbb{N}}$ necessarily converge pointwise a.e. ? (Proof or counterexample.)

5*. (a) Prove the inequality

$$\int_{\mathbb{R}^3} |f(x, y)g(y, z)h(z, x)| d(x, y, z) \leq \|f\|_2 \|g\|_2 \|h\|_2$$

for $f, g, h \in L^2(\mathbb{R}^2)$.
(b) Let $E \subset \mathbb{R}^3$ be a measurable set and suppose that the projections

$$E_1 = \{(y, z) : (x, y, z) \in E\},$$
$$E_2 = \{(x, z) : (x, y, z) \in E\},$$
$$E_3 = \{(x, y) : (x, y, z) \in E\}$$

are measurable subsets of $\mathbb{R}^2$. Use (a) to derive an upper bound on the measure of E in terms of the measures of E_1, E_2, E_3.