Math 629, Spring 2019 – Homework 7.
Due Monday, March 25.

(Problems with an asterisk (*) are optional.)

1. Suppose that $f \in L^1(\mathbb{R}^d)$ and $\lambda_1, \ldots, \lambda_d > 0$. Define

$$g(x) = f(\lambda_1^{-1}x_1, \ldots, \lambda_d^{-1}x_d)$$

Show that $g \in L^1(\mathbb{R}^d)$ and that

$$\int g = \Big(\prod_{j=1}^{d} \lambda_j\Big) \int f.$$

2. Let $f, g \in L^1(\mathbb{R}^d)$. The *convolution* of f and g is the function $f * g$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$$

(a) Show that $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^d$. That is, show that for a.e. $x \in \mathbb{R}^d$, the function $y \mapsto f(x-y)g(y)$ is integrable (in particular, show that it is measurable).
(b) Verify that the following properties hold for every $f, g, h \in L^1(\mathbb{R}^d)$:

$$f * g = g * f, \quad (f * g) * h = f * (g * h),$$
$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

(c) Show that there does not exist a function $\delta \in L^1(\mathbb{R}^d)$ such that

$$f * \delta = f \text{ for all } f \in L^1(\mathbb{R}^d)$$

(d*) Let $E_1, E_2 \subset \mathbb{R}^d$ be measurable and $m(E_1) > 0, m(E_2) > 0$. Show that $E_1 + E_2$ contains an open ball. *Hint:* Consider the function $\mathbf{1}_{E_1} * \mathbf{1}_{E_2}$.

3. Let $f \in L^1(\mathbb{R}^d)$. Define the *Fourier transform* of f by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx \quad (\xi \in \mathbb{R}^d)$$

(Here $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$.) Observe that $\widehat{f}(\xi)$ is well-defined, since $|e^{-2\pi i x \cdot \xi}| = 1$ for every $x, \xi \in \mathbb{R}^d$, so $x \mapsto f(x)e^{-2\pi i x \cdot \xi}$ is integrable for every $\xi \in \mathbb{R}^d$.
(a) Show that $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ is a continuous function and $\|\widehat{f}\|_\infty \leq \|f\|_1$.
(b) Show that for every $f, g \in L^1(\mathbb{R}^d)$ we have

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

(Turn the page.)

3 (continued). (c) Define *translation* of f by $T_y f(x) = f(x-y)$ for $y \in \mathbb{R}^d$, *modulation* of f by $M_\xi f(x) = e^{2\pi i x\cdot\xi} f(x)$ for $\xi \in \mathbb{R}^d$ and *dilation* of f by $D^p_\delta f(x) = \delta^{\frac{d}{p}} f(\delta x)$ for $\delta > 0$ and $p \in [1,\infty]$. Prove the following properties:

$$\widehat{T_y f} = M_{-y}\widehat{f}, \quad \widehat{M_\xi f} = T_\xi \widehat{f}, \quad \widehat{D^p_\delta f} = D^{p'}_{\delta^{-1}}\widehat{f}$$

(Here $\frac{1}{p} + \frac{1}{p'} = 1$.)
(d) Let $g(x) = e^{-\pi|x|^2}$. Prove that $g \in L^1(\mathbb{R}^d)$ and $\widehat{g} = g$.
Hint: First prove it for $d = 1$ and then use Fubini.

4. Let $f \in L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$. Show that

$$\|f\|_p = \Big(\int_0^\infty p\lambda^{p-1} m(\{|f| > \lambda\}) d\lambda\Big)^{1/p}.$$

5*. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and $g \in L^1(\mathbb{R}^d)$ real-valued. Let $E \subset \mathbb{R}^d$ measurable with $m(E) = 1$. Prove that

$$f\Big(\int_E g\Big) \le \int_E f \circ g$$

Honors problem 4. Let $E \subset \mathbb{R}^d$ be a measurable set with $m(E) = 1$. Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence of integrable real-valued functions supported on E such that for all $k, \ell \in \mathbb{N}$ and $\lambda, \lambda' \in \mathbb{R}$ we have

(1) $m(\{f_k > \lambda\}) = m(\{f_\ell > \lambda\})$
(2) $m(\{f_k > \lambda, f_\ell > \lambda'\}) = m(\{f_k > \lambda\}) m(\{f_\ell > \lambda'\})$
(3) $\int f_k = 0$
(4) $\|f_k\|_2 = 1$

For $n \in \mathbb{N}$ define the function

$$A_n = \frac{1}{n}\sum_{k=1}^n f_k.$$

Show that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} m(\{|A_n| > \varepsilon\}) = 0.$$

Hint: First show that $\int f_k f_\ell = 0$ for every $k, \ell \in \mathbb{N}$. Then compute $\|A_n\|_2$ and use Chebyshev's inequality (see Homework sheet 5).