Math 629, Spring 2019 – Homework 8.
Due Monday, April 1.

(Problems with an asterisk (*) are optional.)

1. Prove that the following classes of functions are dense[1] in $L^1(\mathbb{R}^d)$:
(a) Simple functions
(b) Step functions
(c) Continuous functions with compact support
Hints: (a) Use a pointwise approximation by simple functions. (b) By (a) it suffices to approximate the characteristic function of a measurable set of finite measure by step functions. (c) By (b) it suffices to approximate the characteristic function of a cube.

2. For $f \in L^1_{\text{loc}}(\mathbb{R})$, define the (centered) Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x-t)| dt, \quad (x \in \mathbb{R})$$

(a) Fix real numbers $a < b$. Compute $M\mathbf{1}_{[a,b]}$ explicitly.
(b) Suppose that $Mf \in L^1(\mathbb{R})$ for some $f \in L^1_{\text{loc}}(\mathbb{R})$. Prove that $f = 0$ a.e.

3. Let $E \subset [0,1]$ be measurable and $\alpha \in (0,1)$. Assume that for every interval $I \subset [0,1]$ we have $m(E \cap I) \geq \alpha \cdot m(I)$. Prove that $m(E) = 1$.

4. Let $E \subset \mathbb{R}$ be a measurable set with $m(E) > 0$ and k a positive integer. Show that there exist $x \in \mathbb{R}, y > 0$ such that the points

$$x, x+y, x+2y, \dots, x+ky$$

are all contained in E. *Hint:* Use the Lebesgue density theorem.

5*. Let φ be a non-negative integrable function on $\mathbb{R}$ which is even (that is, $\varphi(x) = \varphi(-x)$ for all $x \in \mathbb{R}$) and non-increasing on $[0, \infty)$. For $t > 0$ we let $\varphi_t(x) = t^{-1}\varphi(t^{-1}x)$. Prove that for a.e. $x \in \mathbb{R}$ and every integrable function f we have

$$\sup_{t>0} |f * \varphi_t(x)| \leq \|\varphi\|_1 Mf(x),$$

where Mf denotes the centered Hardy-Littlewood maximal function as defined in Problem 1. *Hint:* First prove it if φ is a simple function.

6*. Do Problem 8 on the strong maximal function in Chapter 3.6 in Stein-Shakarchi "Real Analysis".

[1] Recall that a subset A of a normed vector space X is called *dense* if $\overline{A} = X$, i.e. every $x \in X$ is the limit (wrt. the norm in X) of a sequence in A.