Math 629, Spring 2019 – Homework 9.
Due Monday, April 8.

(Problems with an asterisk (*) are optional.)

1. Let $K \in L^1(\mathbb{R}^d)$ be bounded and supported on a bounded set with $\int K = 1$. Define $K_\delta(x) = \delta^{-d} K(\delta^{-1}x)$ for every $\delta > 0$. Show that $(K_\delta)_\delta$ is an approximation of identity in the sense defined in the lecture.

2. Suppose that $(K_\delta)_{\delta>0}$ is a family of integrable functions such that there exists a constant $C \in (0, \infty)$ such that $\int K_\delta = 1$, $\int |K_\delta| \le C$ for every $\delta > 0$ and for every $\varepsilon > 0$ we have

$$\lim_{\delta \to 0+} \int_{|x| \ge \varepsilon} |K_\delta(x)| dx = 0.$$

Let $p \in [1, \infty)$. Prove that for every $f \in L^p(\mathbb{R}^d)$ we have $f * K_\delta \to f$ in $L^p(\mathbb{R}^d)$ as $\delta \to 0+$. *Hint:* Adapt the proof seen in the lecture.

3. Let $E \subset \mathbb{R}$ be measurable with $m(E) > 0$. Does there exist a sequence $(s_n)_{n \in \mathbb{N}}$ such that the complement of $\bigcup_{n \in \mathbb{N}} (s_n + E)$ has measure zero? (Prove or disprove.)

4. (i) Let $f \ge 0$ be a bounded function and $E \subset \mathbb{R}^d$ have finite measure. Prove that there exists $R > 0$ such that for all $0 < r < R$ we have

$$\int_E f \le 2 \int_E \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} f \Big) dx$$

(ii) Let $f \ge 0$, $R \ge 0$ and B_R a ball of radius R in $\mathbb{R}^d$. Show that for every $0 < r < R$ we have

$$\int_{B_R} f \le C \int_{B_R} \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} f \Big) dx,$$

where C is a constant only depending on d (but not on B_R and f) and $B(x,r) = \{y \in \mathbb{R}^d : |x - y| \le r\}$.

(Turn the page.)

Honors problem 5. In this exercise we introduce a variant of the Hardy-Littlewood maximal function and use it to give an alternative proof of the weak (1,1) bound for the Hardy-Littlewood maximal function. Fix a real number s. By $\mathcal{D}_s$ we denote the collection of intervals of the form $[2^k(\ell + s), 2^k(\ell + s + 1))$ with $k, \ell \in \mathbb{Z}$. These are called *dyadic intervals* (with shift s). Observe that every two dyadic intervals $I, J \subset \mathcal{D}_s$ have the property that they are either disjoint or contained in each other. Given $f \in L^1(\mathbb{R}^d)$, define the *dyadic maximal function*

$$M_s f(x) = \sup_{I \in \mathcal{D}_s, x \in I} \frac{1}{|I|} \int_I |f| \quad (x \in \mathbb{R})$$

(i) Show that for every $\lambda > 0$, the set $\{M_0 f > \lambda\} \subset \mathbb{R}$ can be written as a union over pairwise disjoint dyadic intervals in $\mathcal{D}_0$.

(ii) Prove that there exists a constant $C > 0$ such that for every $\lambda > 0$ and every $f \in L^1(\mathbb{R})$ we have[1]

$$|\{M_0 f > \lambda\}| \leq C\lambda^{-1} \|f\|_1.$$

(iii) Observe that (ii) also holds with M_s in place of M_0 for every $s \in \mathbb{R}$.

(iv) Show that there exists a constant $c > 0$ such that for every interval $I \subset \mathbb{R}$ there exists an interval $J \in \mathcal{D}_0 \cup \mathcal{D}_{1/3} \cup \mathcal{D}_{2/3}$ such that $I \subset J$ and $|J| \leq c|I|$.

(v) Let M denote the Hardy-Littlewood maximal function as defined in the lecture. Show that there exists $c > 0$ such that

$$Mf \leq c(M_0 f + M_{1/3} f + M_{2/3} f)$$

In particular, $m(\{Mf > \lambda\}) \leq c' \lambda^{-1} \|f\|_1$ for every $\lambda > 0$.

[1] You are not allowed to use the weak (1,1) bound for the Hardy-Littlewood maximal function seen in the lecture!