CS/Math 514, Fall 2019 – Numerical Analysis (Roos)
Homework assignment 1.
Due Monday, September 16.

Problems with an asterisk (*) are optional.

1. (i) Implement a function `simple(f,x0,N)` that takes in a function f, a starting point x_0 and uses at most N iterations of the simple fixed point iteration $x_k = g(x_{k-1})$ where $g(x) = f(x) + x$ to compute a solution to $f(x) = 0$. To test your program, compute the solution to $\cos(x) - x = 0$ up to at least 6 significant decimal digits.

(ii) Implement a function `newton(f,fderiv,x0,N)` that takes in a function f, its derivative f', a starting point x_0 and uses Newton's method with at most N iterations to compute a solution to $f(x) = 0$. To test your program, compute the (unique) solution to each of the following equations up to 6 significant decimal digits:

$$(1)\ \cos(x) - x = 0 \qquad (2)\ e^{-x} - x^2 + 5 = 0 \qquad (3)\ e^x = \frac{10}{1+10x^2}$$

(iii) Compare these two methods for computing $\cos(x) - x = 0$ as follows: for both of your implementations, `simple` and `newton`, plot the (negative) logarithmic error[1] in the kth iteration against k (with $N = 100$, say). Do this for each of the initial values $x_0 = 1, 4, 10, 100$. Explain the results in light of the theoretical results seen in class.

(iv*) For each of the equations in (ii) plot the function f you used to solve it. Visualize the corresponding Newton iterates; for example by appropriately connecting the points $(x_k, 0)$, $(x_k, f(x_k))$ for $k = 0, 1, \ldots$ with line segments.

Practical hints: In (i), (ii) it makes sense to terminate computation once the solution no longer improves (i.e. difference between subsequent iterates is zero[2]). You may implement other parameters (such as target precision, verbosity settings etc.) as you see fit. In (iii), to avoid division-by-zero errors when computing $\log|x|$, it is reasonable to compute $\log(|x| + \epsilon)$ instead, where ϵ is the machine epsilon[3] (e.g. in Python/NumPy one may use `eps = np.finfo(np.double).eps`).

2. Let $c \in (0, 1)$ and define $g(x) = \frac{1}{2}(x^2 + c)$. The function g has two fixed points, $\xi_1 < 1$ and $\xi_2 > 1$. For each starting value $x_0 \in \mathbb{R}$, determine the limiting behavior of the fixed point iteration $x_{k+1} = g(x_k)$. That is, prove or disprove existence of the limit $\lim_{k\to\infty} x_k$ and if it exists, determine its value. *Hint:* Draw a picture or do numerical experiments to make conjectures and then prove them.

Turn the page.

[1] Let ξ be the solution and x_k the kth iterate, then plot $-\log_{10}(|\xi - x_k|)$. This is roughly the number of correct decimal digits (up to rounding convention).

[2] Note this does not mean that the computed solution is exact!

[3] This is the smallest floating point number ϵ such that $1 \oplus \epsilon > 1$ (where $\oplus$ is addition of floating point numbers).

3. Consider the equation

$$x + e^{-cx^2}\cos(x) = 0,$$

where $c > 0$ is a parameter.
(i) Prove that the equation has a unique solution $x \in \mathbb{R}$ for every $c > 0$.
(ii) Use Newton's method to compute the (unique) solution to the equation for each of the values $c = 1, 10, 100$ up to at least 6 significant decimal digits. (Try different values for x_0, including $x_0 = 0$.)
(iii) Describe and explain what happened when you tried $x_0 = 0$ in (ii).

4. Let $a > 0$. Consider the iteration

$$x_{k+1} = \tfrac{1}{2}(x_k + \tfrac{a}{x_k})$$

with $x_0 > 0$.
(i) Prove that $\xi = \lim_{k\to\infty} x_k$ exists and determine its value.
Hint: Show that the sequence $(x_k)_{k\geq 1}$ is decreasing and bounded.
(ii) Show that $(x_k)_k$ converges at least quadratically.
Hint: Find a recursive formula for the error $e_k = x_k - \xi$.

5*.
(i) As in Problem 1 above, implement the secant method as a function `secant(f, x0, x1, N)` and solve the equations given in Problem 1 (ii).
(ii) Similarly, implement the bisection method as a function `bisect(f, a, b, N)`.
(iii) Create plots as in Problem 1 (iii), (iv) to compare the Newton, secant and bisection methods. Experiment with different equations and starting points.