Math 522, Fall 2019 – Analysis II (Roos)

Homework assignment 1. Due Monday, September 16.

Problems with an asterisk (*) are optional.

1. Prove or disprove convergence for each of the following series (a and b are real parameters and convergence may depend on their values).

$$\text{(i)} \sum_{n=2}^{\infty} \frac{1}{n^a \log(n)^b} \qquad \text{(ii)} \sum_{n=3}^{\infty} (\log n)^{a \frac{\log n}{\log\log n}} \qquad \text{(iii)} \sum_{n=1}^{\infty} \left(e^{1/n} - \tfrac{n+1}{n}\right)$$

$$\text{(iv)} \sum_{n=2}^{\infty} 2^{-(\log(n))^a} \qquad \text{(v)} \sum_{n=1}^{\infty} \Big(\sum_{k=0}^{10n} (-1)^k \frac{n^k}{k!} \Big) \qquad \text{(vi*)} \sum_{n=1}^{\infty} \frac{1}{n^2(1-\cos(n))}$$

2. Prove or disprove convergence for each of the following sequences and in case of convergence, determine the limit:

(i) $a_0 = 1$, $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$

(ii) $b_n = n \sum_{k=0}^{\infty} \frac{1}{n^2+k^2}$

(iii) $c_n = \sum_{k=n}^{n^2} \frac{1}{k}$

(iv) $d_n = \prod_{k=2}^{n} \frac{k^2-1}{k^2}$

3. For each of the following series, determine whether it converges uniformly on $\mathbb{R}$ and determine whether it converges uniformly on each closed interval $[a,b] \subset \mathbb{R}$:

$$\text{(i)} \sum_{n=1}^{\infty} \frac{x^n}{n^n} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \quad \text{(iii)} \sum_{n=1}^{\infty} 2^{-n} \tan(\lfloor x \rfloor + 1/n)$$

(Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$.)

4. (i) Give an example of a sequence $(f_n)_n$ of continuously differentiable functions defined on $\mathbb{R}$, uniformly convergent on $\mathbb{R}$ such that the limit $\lim_{n\to\infty} f_n'(x)$ does not exist for any value of $x \in \mathbb{R}$.

(ii) Give an example of a sequence $(f_n)_n$ of continuously differentiable functions defined on $\mathbb{R}$, uniformly convergent on $\mathbb{R}$ to some function f such that f is not differentiable.

5*. Determine the value of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$.