Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 10. Due Wednesday, November 27.

(Problems with an asterisk (*) are optional.)

1. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions (that is, all partial derivatives exist to arbitrary orders and are continuous). Show that for all multiindices $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha (f \cdot g)(x) = \sum_{\beta \in \mathbb{N}_0^n \,:\, \beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha - \beta} g(x)$$

for all $x \in \mathbb{R}^n$, where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \frac{\alpha_1! \cdots \alpha_n!}{\beta_1! \cdots \beta_n! (\alpha_1 - \beta_1)! \cdots (\alpha_n - \beta_n)!}$.

2. Let $E \subset \mathbb{R}^n$ be open, $f : E \to \mathbb{R}$ and $x \in E$. Assume that for y in a neighborhood of 0 we have

$$f(x+y) = \sum_{|\alpha| \leq k} c_\alpha y^\alpha + o(\|y\|^k)$$

as $y \to 0$ and

$$f(x+y) = \sum_{|\alpha| \leq k} \widetilde{c}_\alpha y^\alpha + o(\|y\|^k)$$

as $y \to 0$. Show that $c_\alpha = \widetilde{c}_\alpha$ for all $|\alpha| \leq k$.

3. Let F be a smooth function on $\mathbb{R}^2$ and suppose that the initial value problem $y' = F(t, y)$, $y(t_0) = y_0$ has a unique solution y on the interval $I = [t_0, t_0 + a]$ with y smooth on I. Let $h > 0$ be sufficiently small and define $t_k = t_0 + kh$ for integers $0 \leq k \leq a/h$.
Define a function y_h recursively by setting $y_h(t_0) = y_0$ and

$$y_h(t) = y_h(t_k) + (t - t_k) F(t_k, y_h(t_k))$$

for $t \in (t_k, t_{k+1}]$ for integers $0 \leq k \leq a/h$.
(i) From the proof of Peano's theorem that we saw in class it follows that $y_h \to y$ uniformly on I as $h \to 0$. Prove the following stronger statement: there exists a constant $C > 0$ such that for all $t \in I$ and $h > 0$ sufficiently small,

$$|y(t) - y_h(t)| \leq Ch.$$

Hint: The left hand side is zero if $t = t_0$. Use Taylor expansion to study how the error changes as t increases from t_k to t_{k+1}.
(ii) Let $F(t, y) = \lambda y$ with $\lambda \in \mathbb{R}$ a parameter. Explicitly determine y, y_h and a value for C in (i).

(Please turn the page.)

4*. Let us improve the approximation from Problem 3. In the context of Problem 3, define a piecewise linear function y_h^* recursively by setting $y_h^*(t_0) = y_0$ and

$$y_h^*(t) = y_h^*(t_k) + (t - t_k)G(t_k, y_h^*(t_k), h),$$

for $t \in (t_k, t_{k+1}]$ for integers $0 \le k \le a/h$, where

$$G(t, y, h) = \frac{1}{2}(F(t, y) + F(t + h, y + hF(t, y))).$$

Prove that there exists a constant $C > 0$ such that for all $t \in I$ and $h > 0$ sufficiently small,

$$|y(t) - y_h^*(t)| \le Ch^2.$$

5*. Let $f \in C^2(\mathbb{R}^n)$ and suppose that the Hessian of f is positive definite at every point. Show that $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is an injective map.