Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 2. Due Monday, September 23.

(Problems with an asterisk (*) are optional.)

1. Let (X, d) be a metric space and $A \subset X$ a subset.
(i) Show that A is totally bounded if and only if $\overline{A}$ is totally bounded.
(ii) Assume that X is complete. Show that A is totally bounded if and only if A is relatively compact. Which direction is still always true if X is not complete?

2. Recall that ℓ^∞ is the metric space of bounded sequences of complex numbers equipped with the supremum metric $d(a, b) = \sup_{n\in\mathbb{N}} |a_n - b_n|$. Let $s \in \ell^\infty$ be a sequence of non–negative real numbers that converges to zero. Let

$$\mathcal{F} = \{a \in \ell^\infty \, : \, |a_n| \leq s_n \text{ for all } n\}.$$

Prove that $\mathcal{F} \subset \ell^\infty$ is compact.

3. For each of the following subsets of $C([0, 1])$ prove or disprove compactness:
(i) $A_1 = \{f \in C([0, 1]) \, : \, \max_{x\in[0,1]} |f(x)| \leq 1\}$,
(ii) $A_3 = A_1 \cap \{p \, : \, p \text{ polynomial of degree } \leq d\}$ (where $d \in \mathbb{N}$ is given)
(iii) $A_4 = A_1 \cap \{f \, : \, f \text{ is a power series with infinite radius of convergence}\}$

4. Let $\mathcal{F} \subset C([a, b])$ be a bounded set. Assume that there exists a function $\omega : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to 0+} \omega(t) = \omega(0) = 0.$$

and for all $x, y \in [a, b]$, $f \in \mathcal{F}$,

$$|f(x) - f(y)| \leq \omega(|x - y|).$$

Show that $\mathcal{F} \subset C([a, b])$ is relatively compact.

5. Consider $\mathcal{F} = \{f_N \, : \, N \in \mathbb{N}\}$ with

$$f_N(x) = \sum_{n=0}^{N} b^{-n\alpha} \sin(b^n x),$$

where $0 < \alpha < 1$ and $b > 1$ are fixed.
(i) Show that $\mathcal{F}$ is relatively compact in $C([0, 1])$.
(ii) Show that $\mathcal{F}'$ is not a bounded subset of $C([0, 1])$.
(iii*) Show that there exists $c > 0$ such that for all $x, y \in \mathbb{R}$ and $N \in \mathbb{N}$ we have

$$|f_N(x) - f_N(y)| \leq c|x - y|^\alpha.$$

Turn the page.

6*. Let X be a metric space. Assume that for every continuous function $f : X \to \mathbb{C}$ there exists a constant $C_f > 0$ such that $|f(x)| \leq C_f$ for all $x \in X$. Show that X is compact. *Hint:* Assume that X is not sequentially compact and construct an unbounded continuous function on X.

Honors problem 1. For $1 \leq p < \infty$ we denote by ℓ^p the space of sequences $(a_n)_n$ of complex numbers such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Define a metric on ℓ^p by

$$d(a,b) = \left(\sum_{n \in \mathbb{N}} |a_n - b_n|^p \right)^{1/p}.$$

The purpose of this exercise is to prove a theorem of Fréchet that characterizes compactness in ℓ^p. Let $\mathcal{F} \subset \ell^p$.

(i) Assume that $\mathcal{F}$ is bounded and *equisummable* in the following sense: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |a_n|^p < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Then show that $\mathcal{F}$ is totally bounded.

(ii) Conversely, assume that $\mathcal{F}$ is totally bounded. Then show that it is equisummable in the above sense.