Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 3. Due Monday, September 30.

(Problems with an asterisk (*) are optional.)

1. Let d be a positive integer and $f \in C([a,b])$. Denote by P_d the set of polynomials with real coefficients of degree $\leq d$. Prove that there exists a polynomial $p_* \in P_d$ such that $\|f - p_*\|_\infty = \inf_{p \in P_d} \|f - p\|_\infty$. *Hint:* Use compactness.

2. Determine explicitly a sequence of polynomials $(p_n)_n$ that converges uniformly to the function $x \mapsto |x|$ on $[-1, 1]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function (i.e. derivatives of all orders exist). Assume that there exist $A > 0, R > 0$ such that

$$|f^{(n)}(x)| \leq A^n n!$$

for $|x| < R$. Show that there exists $r > 0$ such that for every $|x| < r$ we have that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

(That is, prove that the series on the right hand side converges and that the limit is $f(x)$.)

4. Define a sequence of polynomials $(T_n)_n$ by $T_0(x) = 1$, $T_1(x) = x$ and the recurrence relation $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.
(i) Show that $T_n(x) = \cos(nt)$ if $x = \cos(t)$.
Hint: Use that $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$ for all $a, b \in \mathbb{C}$.
(ii) Compute

$$\int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}}$$

for all non-negative integers n, m.
(iii) Prove that $|T_n(x)| \leq 1$ for $x \in [-1, 1]$ and determine when there is equality.

Turn the page.

5*. Let f be smooth on $[0,1]$ (that is, f is arbitrarily often differentiable).
(i) Let p be a polynomial such that $|f'(x)-p(x)| \le \varepsilon$ for all $x \in [0,1]$. Construct a polynomial q such that $|f(x)-q(x)| \le \varepsilon$ for all $x \in [0,1]$.
(ii) Prove that there exists a sequence of polynomials $(p_n)_n$ such that $(p_n^{(k)})_n$ converges uniformly on $[0,1]$ to $f^{(k)}$ for all $k = 0,1,2,\dots$.

Honors Problem 2. Let f be a continuous function on $[0,1]$ and N a positive integer. Define $x_k = \frac{k}{N}$ for $k = 0,\dots,N$. Define

$$L_N(x) = \sum_{j=0}^{N} f(x_k) \prod_{j=0, j\neq k}^{N} \frac{x-x_j}{x_k-x_j}.$$

(i) Show that $f(x_k) = L_N(x_k)$ for all $k = 0,\dots,N$ and that L_N is the unique polynomial of degree $\le N$ with this property.
(ii) Suppose $f \in C^{N+1}([0,1])$. Show that for every $x \in [0,1]$ there exists $\xi \in [0,1]$ such that

$$f(x) - L_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{k=0}^{N} (x-x_k).$$

(iii) Show that L_N does not necessarily converge to f uniformly on $[0,1]$. (Find a counterexample.)
(iv*) Suppose f is given by a power series with infinite convergence radius. Does L_N necessarily converge to f uniformly on $[0,1]$?