Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 4. Due Monday, October 7.

(Problems with an asterisk (*) are optional.)

1. Let $(f_n)_n$ be a sequence of continuous functions on $[0,1]$ and f a continuous function on $[0,1]$. Assume that $\int_0^1 |f_n - f| \to 0$. Does it follow that $f_n(x) \to f(x)$ for some $x \in [0,1]$? Give a proof or counterexample.

2. (i) Let $(a_k)_k$ be a sequence of real or complex numbers with limit L. Prove that

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = L$$

Given the sequence a_k, form the partial sums $s_n = \sum_{k=1}^n a_k$ and let

$$\sigma_N = \frac{s_1 + \cdots + s_N}{N}.$$

σ_N is called the Nth Cesàro mean of the sequence s_k or the Nth Cesàro sum of the series $\sum_{k=1}^\infty a_k$. If σ_N converges to a limit S we say that the series $\sum_{k=1}^\infty a_k$ is Cesàro summable to S.

(ii) Prove that if $\sum_{k=1}^\infty a_k$ is summable to S (i.e. by definition converges with sum S) then $\sum_{k=1}^\infty a_k$ is Cesàro summable to S.

(iii) Prove that the sum $\sum_{k=1}^\infty (-1)^{k-1}$ does not converge but is Cesàro summable to some limit S and determine S.

3. Show that each of the following is an orthonormal systems on $[0,1]$ (where $n = 1, 2, \dots$):
(i) $\phi_n(x) = \sqrt{2}\cos(2\pi n x)$
(ii) $\phi_n(x) = \sqrt{2}\sin(2\pi(n + \frac{1}{2})x)$
(iii) $\phi_n(x) = \operatorname{sgn}(\sin(2^n \pi x))$

4. Let $f \in C([0,1])$ and $\mathcal{A} \subset C([0,1])$ dense. Suppose that

$$\int_0^1 f(x)\overline{a(x)}dx = 0$$

for all $a \in \mathcal{A}$. Show that $f = 0$.
Hint: Show that $\int_0^1 |f(x)|^2 dx = 0$.

Turn the page.

5*. Define $p_n(x) = \frac{d^n}{dx^n}[(1-x^2)^n]$ for $n = 0, 1, \ldots$ and

$$\phi_n(x) = p_n(x) \cdot \Big(\int_{-1}^{1} p_n(t)^2 dt \Big)^{-1/2}.$$

Show that $(\phi_n)_{n=0,1,\ldots}$ is a complete orthonormal system on $[-1, 1]$.

6*. Fix a function $0 \neq w \in C([0,1])$ with $w(x) \geq 0$ for all $x \in [0,1]$. Prove that there exists a sequence of real-valued polynomials $(p_n)_n$ such that

$$\int_0^1 p_n(x) p_m(x) w(x) dx = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m \end{cases}$$

for all non-negative integers n, m.