Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 5. Due Monday, October 14.

(Problems with an asterisk (*) are optional.)

1. Recall the functions $r_n(x) = \operatorname{sgn}(\sin(2^n \pi x))$ from the previous homework sheet.
(i) Show that every r_n for $n \geq 1$ can be written as a finite linear combination of Haar functions and determine the coefficients of this linear combination.
(ii) Show that the orthonormal system on $[0, 1]$ given by $(r_n)_n$ is not complete.

2. Show that there exists no continuous 1-periodic function g such that $f * g = f$ holds for all continuous 1-periodic functions f.
(Here $(f * g)(x) = \int_0^1 f(x - y)g(y)dy$.)

3. Let f be the 1-periodic function such that $f(x) = |x|$ for $x \in [-1/2, 1/2]$. Determine explicitly a sequence of trigonometric polynomials $(p_N)_N$ such that $p_N \to f$ uniformly as $N \to \infty$.

4. Let f, g be continuous, 1-periodic functions. Recall that for $n \in \mathbb{Z}$, the nth *Fourier coefficient* of f is defined by $\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi itn}dt$.
(i) Show that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$.
(ii) Show that $\widehat{f \cdot g}(n) = \sum_{m \in \mathbb{Z}} \widehat{f}(n - m)\widehat{g}(m)$.
(iii) If f is continuously differentiable, prove that $\widehat{f'}(n) = 2\pi in\widehat{f}(n)$.
(iv) Let $y \in \mathbb{R}$ and set $f_y(x) = f(x + y)$. Show that $\widehat{f_y}(n) = e^{2\pi iny}\widehat{f}(n)$.
(v) Let $m \in \mathbb{Z}$, $m \neq 0$ and set $f_m(x) = f(mx)$. Show that $\widehat{f_m}(n)$ equals $\widehat{f}(\frac{n}{m})$ if m divides n and zero otherwise.

5*. Let f be 1-periodic and k times continuously differentiable. Prove that there exists a constant $c > 0$ such that

$$|\widehat{f}(n)| \leq c|n|^{-k} \quad \text{for all } n \in \mathbb{Z}.$$

Hint: What can you say about the Fourier coefficients of $f^{(k)}$?

Turn the page.

6*. Let f be 1-periodic and continuous.
(i) Suppose that $\widehat{f}(n) = -\widehat{f}(-n) \geq 0$ holds for all $n \geq 0$. Prove that

$$\sum_{n=1}^{\infty} \frac{\widehat{f}(n)}{n} < \infty.$$

(ii) Show that there does not exist a 1-periodic continuous function f such that

$$\widehat{f}(n) = \frac{\operatorname{sgn}(n)}{\log |n|} \quad \text{for all } |n| \geq 2.$$

Here $\operatorname{sgn}(n) = 1$ if $n > 0$ and $\operatorname{sgn}(n) = -1$ if $n < 0$.