Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 6. Due Monday, October 21.

(Problems with an asterisk (*) are optional.)

1. Let K be a compact metric space. Show that if $\mathcal{A} \subset C(K)$ does not separate points or does not vanish nowhere, then $\mathcal{A}$ is not dense.

2. Let K be a finite set and $\mathcal{A}$ a family of functions on K that is a self-adjoint algebra, separates points and vanishes nowhere. Give a purely algebraic proof that $\mathcal{A}$ must then already contain every function on K. (That means your proof is not allowed to use the concept of an inequality. In particular, you are not allowed to use any facts about metric spaces such as the Stone-Weierstrass theorem.) *Hint:* Take a close look at the proof of Stone-Weierstrass.

3. Give a rigorous proof of the identity

$$-\frac{1}{2} = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(n)}{n}.$$

4. Show that there exists a constant $c > 0$ such that

$$\int_0^1 |D_N(x)| dx \geq c \log(2 + N)$$

holds for all $N = 0, 1, \dots$.

5*. Give an alternative proof of Weierstrass' theorem by using Fejér's theorem and then approximating the resulting trigonometric polynomials by truncated Taylor expansions.

6*. Suppose that f is a 1-periodic function such that there exists $c > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x) - f(y)| \leq c|x - y|^\alpha$$

holds for all $x, y \in \mathbb{R}$. Show that the sequence of partial sums $S_N f(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi i n x}$ converges uniformly to f as $N \to \infty$.

(Turn the page.)

Honors Problem 3. Let $\sigma(t) = e^t$ for $t \in \mathbb{R}$. Fix $n \in \mathbb{N}$ and let $K \subset \mathbb{R}^n$ be a compact set. As usual, let $C(K)$ denote the space of real-valued continuous functions on K. Define a class of functions $\mathcal{N} \subset C(K)$ by saying that $\mu \in \mathcal{N}$ iff there exist $m \in \mathbb{N}$, $W \in \mathbb{R}^{m\times n}$, $v, b \in \mathbb{R}^m$ such that

$$\mu(x) = \sum_{i=1}^{m} \sigma((Wx)_i + b_i)v_i \text{ for all } x \in K.$$

Prove that $\mathcal{N}$ is dense in $C(K)$.

Remark. The figure below visualizes the flow of information from the input vector x to the output $\mu(x)$. The definition of this class of functions is motivated by biology.

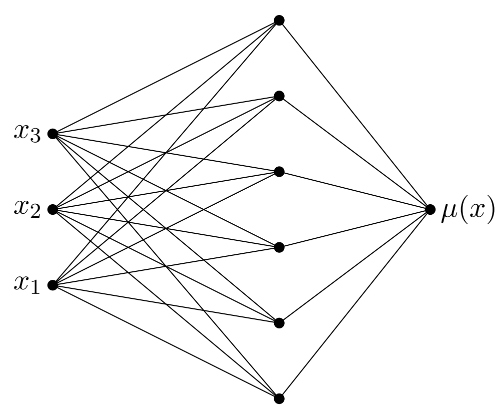

FIGURE 1. Visualization of μ when $n = 3$ and $m = 6$.