Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 7. Due Monday, November 4.

(Problems with an asterisk (*) are optional.)

1. Let X denote the vector space consisting of sequences of complex numbers $(a_n)_{n=1,2,\dots}$ such that $\sum_{n=1}^\infty |a_n| < \infty$. We define $\|a\|_1 = \sum_{n=1}^\infty |a_n|$, $\|a\|_2 = (\sum_{n=1}^\infty |a_n|^2)^{1/2}$, $\|a\|_\infty = \sup_{n=1,2,\dots} |a_n|$. Each of these three defines a norm on X. Prove that no two (different ones) of them are equivalent norms.

2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on a normed vector space X.
(i) Show that a sequence $(x_n)_n \subset X$ converges with respect to $\|\cdot\|_1$ if and only if it converges with respect to $\|\cdot\|_2$.
(ii) Show that a set $U \subset X$ is open with respect to $\|\cdot\|_1$ if and only if it is open with respect to $\|\cdot\|_2$.

3. Let X be the set of continuously differentiable functions on $[0,1]$ and Y the set of continuous functions on $[0,1]$. We consider X and Y as normed vector spaces with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Define a linear map $T : X \to Y$ by $Tf = f'$. Show that T is not bounded.

4. Let $A \in \mathbb{R}^{m\times n}$. For $x \in \mathbb{R}^n$ we define $\|x\|_1 = \sum_{i=1}^n |x_i|$.
(i) Determine the value of $\|A\|_{1\to1} = \sup_{\|x\|_1=1} \|Ax\|_1$ (that is, find a formula for $\|A\|_{1\to1}$ involving only finitely many computations in terms of the entries of A).
(ii) Do the same for $\|A\|_{1\to\infty} = \sup_{\|x\|_1=1} \|Ax\|_\infty$ and $\|A\|_{\infty\to1} = \sup_{\|x\|_\infty=1} \|Ax\|_1$.

5. Let $\mathbb{R}^{n\times n}$ denote the space of real $n \times n$ matrices equipped with the matrix norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$. Define

$$F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n},\ A \longmapsto A^2.$$

Show that F is Fréchet differentiable and compute $DF\,|_A$.

6*. Let $X = C([0,1])$ be the Banach space of continuous functions on $[0,1]$ (with the supremum norm) and define a map $F : X \to X$ by

$$F(f)(s) = \int_0^s \cos(f(t)^2)dt,\ \ s \in [0,1].$$

(i) Show that F is Fréchet differentiable and compute the Fréchet derivative $DF\,|_f$ for each $f \in X$.
(ii) Show that $FX = \{F(f) : f \in X\} \subset X$ is relatively compact.

(Turn the page.)

7*. Let $A \in \mathbb{R}^{n\times n}$. Define $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$ (Euclidean norm) and $\|A\|_{2\to 2} = \sup_{\|x\|_2=1} \|Ax\|_2$. Observe that AA^T is a symmetric $n \times n$ matrix and hence has only non-negative eigenvalues. Denote the largest eigenvalue of AA^T by ρ. Prove that $\|A\|_{2\to 2} = \sqrt{\rho}$. *Hint:* First consider the case that A is symmetric. Use that symmetric matrices are orthogonally diagonalizable.