Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 8. Due Monday, November 11.

(Problems with an asterisk (*) are optional.)

1. Let $U \subseteq \mathbb{R}^n$ be open and convex and $f : U \to \mathbb{R}$ differentiable such that $\partial_1 f(x) = 0$ for all $x \in U$.
(i) Show that the value of $f(x)$ for $x = (x_1, \ldots, x_n) \in U$ does not depend on x_1.
(ii) Does (i) still hold if we assume that U is connected instead of convex? Give a proof or counterexample.

2. Show that there exists a unique $(x, y) \in \mathbb{R}^2$ such that $\cos(\sin(x)) = y$ and $\sin(\cos(y)) = x$.

3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous of degree* $\alpha \in \mathbb{R}$ if $f(\lambda x) = \lambda^\alpha f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Suppose that f is differentiable. Then show that f is homogeneous of degree α if and only if

$$\sum_{i=1}^n x_i \partial_i f(x) = \alpha f(x)$$

for all $x \in \mathbb{R}^n$. *Hint:* Consider the function $g(\lambda) = f(\lambda x) - \lambda^\alpha f(x)$.

4. Prove that there exists $\delta > 0$ such that for all square matrices $A \in \mathbb{R}^{n\times n}$ with $\|A - I\| < \delta$ (where I denotes the identity matrix and $\|\cdot\|$ is some norm on $\mathbb{R}^{n\times n}$) there exists $B \in \mathbb{R}^{n\times n}$ such that $B^2 = A$.

5*. Find a C^1 map $F : \mathbb{R} \to \mathbb{R}^2$ and points $x, y \in \mathbb{R}$ such that there does not exist $\xi \in \mathbb{R}$ such that $F(x) - F(y) = DF|_\xi (x - y)$.

Honors problem 4. Let $L \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ (i.e. all partial derivatives of L up to order 2 exist on $(a, b) \times \mathbb{R} \times \mathbb{R}$ and are continuous on $[a, b] \times \mathbb{R} \times \mathbb{R}$). For $f : [a, b] \to \mathbb{R}$ we define

$$\mathscr{I}(f) = \int_a^b L(t, f(t), f'(t))dt.$$

Let $\mathcal{A} = \{f \in C^2([a, b]) : f(a) = f(b) = 0\}$ and assume that $f_* \in \mathcal{A}$ is such that

$$\mathscr{I}(f_*) = \inf_{f\in\mathcal{A}} \mathscr{I}(f).$$

Prove that

$$\partial_2 L(t, f_*(t), f_*'(t)) - \frac{d}{dt}\partial_3 L(t, f_*(t), f_*'(t)) = 0.$$

Hint: Consider perturbations $g_\varepsilon(t) = f_*(t) + \varepsilon h(t)$ and differentiate in ε.