Math 522, Fall 2019 – Analysis II (Roos)
Homework assignment 9. Due Monday, November 18.

(Problems with an asterisk (*) are optional.)

1. Look at each of the following as an equation to be solved for $x \in \mathbb{R}$ in terms of parameter $y, z \in \mathbb{R}$. Notice that $(x, y, z) = (0, 0, 0)$ is a solution for each of these equations. For each one, prove that it can be solved for x as a C^1-function of y, z in a neighborhood of $(0, 0, 0)$.

(a) $\cos(x)^2 - e^{\sin(xy)^3 + x} = z^2$
(b) $(x^2 + y^3 + z^4)^2 = \sin(x - y + z)$
(c) $x^7 + ye^z x^3 - x^2 + x = \log(1 + y^2 + z^2)$

2. Let $(t_0, y_0) \in \mathbb{R}^2$, $c \in \mathbb{R}$ and define $Y_0(t) = y_0$,

$$Y_n(t) = y_0 + c \int_{t_0}^{t} s Y_{n-1}(s) ds.$$

Compute $Y_n(t)$ and $Y(t) = \lim_{n\to\infty} Y_n(t)$. Which initial value problem does Y solve?

3. Consider the initial value problem

$$\begin{cases} y'(t) = e^{y(t)^2} - \frac{1}{ty(t)}, \\ y(1) = 1. \end{cases}$$

Find an interval $I = (1 - h, 1 + h)$ such that this problem has a unique solution y in I. Give an explicit estimate for h (it does not need to be best possible).

4. Consider the initial value problem

$$\begin{cases} y'(t) = t + \sin(y(t)), \\ y(2) = 1. \end{cases}$$

Find the largest interval $I \subseteq \mathbb{R}$ containing $t_0 = 2$ such that the problem has a unique solutions y in I.

5*. For a function $f : [a, b] \to \mathbb{R}$ define

$$\mathscr{I}(f) = \int_a^b (1 + f'(t)^2)^{1/2} dt.$$

Let $\mathcal{A} = \{f \in C^2([a, b]) : f(a) = c, f(b) = d\}$. Determine $f_* \in \mathcal{A}$ such that

$$\mathscr{I}(f_*) = \inf_{f \in \mathcal{A}} \mathscr{I}(f).$$

What is the geometric meaning of $\mathscr{I}(f)$ and $\inf_{f\in\mathcal{A}} \mathscr{I}(f)$?