MATH 629 – SPRING 2020 – UW MADISON
L^p SPACES

INSTRUCTOR: J. ROOS

Let (X, Σ, μ) be a σ-finite measure space. As a main example we'll use the special case $(\mathbb{R}^d, \Sigma_{\mathcal{L}}, m)$ with $\Sigma_{\mathcal{L}}$ the Lebesgue σ–algebra and m Lebesgue measure.

Let $f : X \to \mathbb{C}$ be a measurable function. Recall from the lecture that we call f *(absolutely) μ–integrable* (or just *integrable* if μ is clear from context) if $\int |f| d\mu < \infty$.

1. The space $L^1(\mu)$. Observe that the space of μ–integrable functions forms a vector space: if f, g are integrable and λ is a scalar, then $f + \lambda g$ is integrable. Can we equip the space of integrable functions with a norm? A natural choice seems to be

$$\|f\|_1 = \int |f| d\mu.$$

Is this a norm? Let's check.

(1) *Triangle inequality:* for f, g integrable,

$$\|f + g\|_1 = \int |f + g| d\mu \leq \int |f| d\mu + \int |g| d\mu = \|f\|_1 + \|g\|_1$$

(2) *Homogeneity:* for f integrable and λ a scalar, we have

$$\|\lambda f\|_1 = \int |\lambda f| d\mu = |\lambda| \int |f| d\mu = |\lambda| \cdot \|f\|_1$$

(3) *Definiteness:* we should have that $\|f\|_1 = 0$ if and only if $f = 0$. This *fails* for trivial reasons: the integral does not detect what happens on μ–nullsets. For example, $f = \mathbf{1}_{\mathbb{Q}^d}$ is a non-zero Lebesgue integrable function with $\|f\|_{L^1(m)} = 0$.

Luckily, there is an easy way out to repair the problem with definiteness. This embraces our general philosophy that *we don't care what happens on a set of measure zero*. Define an *equivalence relation* on the set of integrable functions as follows: two functions f, g are *equivalent*, $f \sim g$, if and only if $f = g$ μ-a.e. (that is, if there exists $E \in \Sigma$ such that $\mu(E) = 0$ and $f(x) = g(x)$ for all $x \notin E$). Letting $\mathcal{I}$ denote the set of μ-integrable functions, we now define $L^1(\mu)$ as the set of equivalence classes of integrable functions with respect to $\sim$:

$$L^1(\mu) = \mathcal{I}/\sim$$

That is, an element of $L^1(\mu)$ is an equivalence class: $[f] \in L^1(\mu)$ with $[f] = \{g \in \mathcal{I} : f \sim g\}$. Alternative notations are $L^1(X)$ or $L^1(X, \mu)$, or even $L^1(X, \Sigma, \mu)$ if we want to be verbose. Write $L^1(\mathbb{R}^d)$ to denote $L^1(m)$ with m the Lebesgue measure on $\mathbb{R}^d$.

Clearly, $L^1(\mu)$ is still a vector space. We define a norm on it by setting

$$\|[f]\|_{L^1(\mu)} = \|f\|_1 = \int |f| d\mu.$$

Note that this is well-defined since $\|f\|_1 = \|g\|_1$ for every $f \sim g$. Also, this norm still satisfies the triangle inequality and is homogeneous. Moreover, it is now also definite because suppose $\|[f]\|_{L^1(\mu)} = 0$. Then $\int |f| d\mu = 0$, so $f = 0$ μ-a.e. That is, $[f] = 0$ (here $0 \in L^1(\mu)$ denotes the equivalence class of the zero function).

Because it is cumbersome to speak of equivalence classes all the time (and it also comes with notational overhead, since we need to write $[f]$ rather than f), we adopt the following standard convention:

Convention. We continue to speak of elements of $L^1(\mu)$ as *functions* and whenever we write $f \in L^1(\mu)$, then f is silently assumed to be some (mostly arbitrary) member of the corresponding equivalence class.

Remarks. 1. *Careful:* As a consequence, it is generally nonsensical to speak of individual values of f: for example, the value $f(0)$ is not well-defined for an L^1 function f. However, it still makes sense to speak of the values of f in an *almost-everywhere-sense.* For instance, it makes sense to say that $f \geq 0$ a.e.

2. Even with this convention in mind, it may still make sense to speak of *regularity properties* of $f \in L^1(\mu)$. For instance, say that X is endowed with a topology. Then when we say that $f \in L^1(\mu)$ is *continuous*, we mean that there exists a (uniquely determined) member in the equivalence class of f which is continuous and we silently choose f to be that function.

3. In the arguments that follow, it is instructive to convince yourself that this convention *does not cause a lack of rigor* by translating the corresponding arguments for yourself into the language of equivalence classes. You should do so until you become convinced that this convention makes sense.

Example 1. Let f be a bounded measurable function supported in a set of finite measure. Then $f \in L^1(\mu)$. Indeed, say $|f| \leq M$ μ-a.e. Then, by monotonicity of the integral,

$$\int |f| d\mu \leq \int M \mathbf{1}_{\mathrm{supp}(f)} d\mu = M \mu(\mathrm{supp}(f)) < \infty.$$

Example 2. Fix $a \in \mathbb{R}$. Consider functions on $\mathbb{R}^d$,

$$f_a(x) = \begin{cases} |x|^a & \text{if} \quad 0 < |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

$$g_a(x) = \begin{cases} |x|^a & \text{if} \quad |x| > 1, \\ 0 & \text{otherwise} \end{cases}$$

Exercise 3. $f_a \in L^1(\mathbb{R}^d)$ if and only if $a > -d$ and $g_a \in L^1(\mathbb{R}^d)$ if and only if $a < -d$.

In particular, this shows that $L^1(\mathbb{R}^d)$ also contains unbounded functions and functions supported on a set of infinite measure.

To decide whether a given function is in L^1 we need to have information on its magnitude and on the size (read: measure) of the sets where it has a certain magnitude. To illustrate this idea, let's look at the following easy criterion to decide whether $f \in L^1(\mu)$.

Lemma 4. *We have $f \in L^1(\mu)$ if and only if*

$$\sum_{k\in\mathbb{Z}} 2^k \mu(\{x \in X \,:\, 2^k \le |f(x)| \le 2^{k+1}\}) < \infty$$

Proof. We already learned that integrals interchange with infinite sums for non-negative quantities. Using this fact, we have

$$\int |f| d\mu = \int \sum_{k\in\mathbb{Z}} |f| \mathbf{1}_{\{2^k \le |f| \le 2^{k+1}\}} d\mu = \sum_{k\in\mathbb{Z}} \int |f| \mathbf{1}_{\{2^k \le |f| \le 2^{k+1}\}} d\mu$$

Combining this with monotonicity of the integral, we observe

$$\int |f| d\mu \le \sum_{k\in\mathbb{Z}} \int 2^{k+1} \mathbf{1}_{\{2^k \le |f| \le 2^{k+1}\}} d\mu = 2 \sum_{k\in\mathbb{Z}} 2^k \mu(\{x \,:\, 2^k \le |f(x)| \le 2^{k+1}\})$$

and similarly,

$$\int |f| d\mu \ge \sum_{k\in\mathbb{Z}} \int 2^{k} \mathbf{1}_{\{2^k \le |f| \le 2^{k+1}\}} d\mu = \sum_{k\in\mathbb{Z}} 2^k \mu(\{x \,:\, 2^k \le |f(x)| \le 2^{k+1}\}).$$

□

Example 2 revisited. Let us decide when f_a is in $L^1(\mathbb{R}^d)$. Clearly it is if $a \ge 0$. So let $a < 0$. We are interested in the measure of the sets

$$E_k = \{2^k \le |x|^a \le 2^{k+1}\} = \{2^{(k+1)/a} \le |x| \le 2^{k/a}\}$$

for $k > 0$. Note that E_k is contained in a cube of side length equal to a constant times $2^{k/a}$ and also contains a cube of side length equal to a constant times $2^{k/a}$. Thus, the measure of E_k is comparable to $2^{dk/a}$. Thus, $f_a \in L^1(\mathbb{R}^d)$ if and only if

$$\sum_{k>0} 2^k 2^{dk/a} < \infty$$

This is the case if and only if $1 + d/a < 0 \Leftrightarrow a > -d$ (keep in mind $a < 0$).

2. L^p spaces. In analysis, it is often desirable to refine the notion of an integrable function in the following way. Let $p \in [1, \infty)$. We say that $f \in L^p(\mu)$ if $|f|^p \in L^1(\mu)$. In other words, if

$$\|f\|_{L^p(\mu)} = \|f\|_p = \Big(\int |f|^p d\mu \Big)^{1/p} < \infty.$$

Example 5. As in Example 2 above we have

$$|x|^a \mathbf{1}_{|x|<1} \in L^p(\mathbb{R}^d) \quad \text{if and only if} \quad a > -d/p,$$
$$|x|^a \mathbf{1}_{|x|>1} \in L^p(\mathbb{R}^d) \quad \text{if and only if} \quad a < -d/p.$$

It is easy to see that $L^p(\mu)$ is a vector space. Also we have that $\|\lambda f\|_p = |\lambda| \cdot \|f\|_p$ for every scalar λ and $f \in L^p(\mu)$. Moreover, if $\|f\|_p = 0$, then $f = 0$ μ-a.e. To prove that $\|\cdot\|_p$ defines a norm it remains to verify the triangle inequality. This is no longer as straightforward as it was in the case $p = 1$. To prove it we need the following inequality that turns out to be an essential tool in the analysis of L^p spaces.

Theorem 6 (Hölder's inequality)**.** *Let $p \in (1, \infty)$ and $p' \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have*

$$\int |fg| d\mu \le \|f\|_p \|g\|_{p'}$$

for all $f \in L^p(\mu)$ and $g \in L^{p'}(\mu)$.

Remark 7. 1. p' is called the *dual exponent* of p.
2. The special case $p = 2$ is called *Cauchy-Schwarz inequality*.

Proof. Without loss of generality we may assume $\|f\|_p = 1$ and $\|g\|_{p'} = 1$ (more precisely, we claim that the general case follows from the case $\|f\|_p = 1$, $\|g\|_{p'} = 1$: indeed for general f, g, use the inequality on $f/\|f\|_p$ and $g/\|g\|_{p'}$).
Now use the following elementary inequality for real numbers (verify it as an exercise; for example using the convexity of the exponential function):

$$|f(x)| \cdot |g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{p'}$$

We obtain

$$\int |fg| d\mu \le \frac{1}{p} \int |f|^p d\mu + \frac{1}{p'} \int |g|^{p'} d\mu = \frac{1}{p}\|f\|_p^p + \frac{1}{p'}\|g\|_{p'}^{p'} = \frac{1}{p} + \frac{1}{p'} = 1.$$

□

Remark 8. The proof also reveals that we have equality in Hölder's inequality precisely if the functions $|f|^p$ and $|g|^{p'}$ are scalar multiples of eachother (almost everywhere, of course).

As an application of Hölder's inequality we can now prove the triangle inequality for L^p norms.

Theorem 9 (Minkowski's inequality)**.** *Let* $p \in (1,\infty)$. *Then*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

for all $f, g \in L^p(\mu)$.

Proof. The key is to use the pointwise inequality

$$|f(x)+g(x)|^p \le |f(x)+g(x)|^{p-1}|f(x)| + |f(x)+g(x)|^{p-1}|g(x)|$$

This gives

$$\|f+g\|_p^p = \int |f+g|^p d\mu \le \int |f+g|^{p-1}|f| d\mu + \int |f+g|^{p-1}|g| d\mu$$

Observe that $p' = \frac{p}{p-1}$, so $|f+g|^{p-1} \in L^{p'}(\mathbb{R}^d)$ with

$$\||f+g|^{p-1}\|_{p'} = \|f+g\|_p^{p-1}.$$

Thus, Hölder's inequality tells us that

$$\|f+g\|_p^p \le \|f+g\|_p^{p-1}\|f\|_p + \|f+g\|_p^{p-1}\|g\|_p.$$

Note that we may assume that $\|f+g\|_p > 0$ since otherwise there is nothing to prove. Thus, dividing by $\|f+g\|_p^{p-1}$ we obtain the claim. □

Exercise 10. Determine under which condition on f, g we have equality in Minkowski's inequality.

This shows that $L^p(\mu)$ is a normed vector space. A natural question is whether it is a Banach space[1]. In view of the fact that a Cauchy sequence in $L^p(\mu)$ does not necessarily converge pointwise almost everywhere (see Exercise 23), this does not seem at all obvious.

Theorem 11 (Riesz-Fischer)**.** $L^p(\mu)$ *is complete for every* $p \in [1,\infty)$. *That is, if* $(f_n)_n \subset L^p(\mu)$ *is a Cauchy sequence with respect to* $\|\cdot\|_p$, *then there exists* $f \in L^p(\mu)$ *such that* $f_n \to f$ *in* $L^p(\mu)$.

Proof. Suppose that $(f_n)_n$ is a Cauchy sequence in $L^p(\mu)$. Then, for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \varepsilon \quad \text{for every } n, m \ge N(\varepsilon).$$

The difficult part is to get a hold of a good candidate for the limiting function f. The plan is to first show that a suitable subsequence actually converges pointwise almost everywhere and then choose f as the pointwise limit of that subsequence. To do this we choose a sequence $(n_k)_k$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p \le 2^{-k} \quad \text{for every } k \in \mathbb{N}$$

[1]Recall that a normed vector space is called *Banach space* if it is complete, i.e. if every Cauchy sequence converges (with respect to the norm).

This is possible by assumption (for instance, $N(2^{-k})$ is a good choice for n_k). We claim that the sequence $(f_{n_k}(x))_k$ converges for almost every x. Define

$$g_\ell(x) = \sum_{k=1}^{\ell} |f_{n_{k+1}}(x) - f_{n_k}(x)| \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

(Note that $g(x)$ may equal ∞.) By Minkowski's inequality and the construction of $(n_k)_k$ we obtain

$$\|g_\ell\|_p \le \sum_{k=1}^{\ell} \|f_{n_{k+1}} - f_{n_k}\|_p \le \sum_{k=1}^{\ell} 2^{-k} < 1.$$

By Fatou's lemma we then have

$$\|g\|_p^p = \int \lim_{\ell\to\infty} |g_\ell|^p d\mu \le \liminf_{\ell\to\infty} \int |g_\ell|^p d\mu = \liminf_{\ell\to\infty} \|g_\ell\|_p^p \le 1.$$

In particular, $g(x) < \infty$ for a.e. x. This proves that the series

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges absolutely for a.e. x. Since the sequence of partial sums of that series is just the sequence $(f_{n_k})_k$, this proves that $f_{n_k}(x) \to f(x)$ for a.e. x as $k \to \infty$. It remains to show that $f_n \to f$ in $L^p(\mathbb{R}^d)$. This is easy now: by Fatou's lemma we have

$$\|f_m - f\|_p^p = \int |f - f_m|^p d\mu \le \liminf_{k\to\infty} \int |f_{n_k} - f_m|^p d\mu \longrightarrow 0 \quad \text{as } m \to \infty,$$

using that $(f_n)_n$ is a Cauchy sequence. □

3. The space L^∞. We say that a measurable function is *essentially bounded* if

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \inf\{M > 0 : |f| \le M \ \mu\text{-a.e}\} < \infty$$

The number $\|f\|_\infty$ is called the *essential supremum* of f. Note that $\|f\|_\infty = \|g\|_\infty$ whenever $f = g$ a.e. The set of (equivalence classes of) essentially bounded functions is called $L^\infty(\mu)$.

Exercise 12. Show that $L^\infty(\mu)$ is a Banach space.

(This is much more straightforward to show than for L^p with $p < \infty$.)

Remark 13. Observe that L^∞ fits nicely into the family of L^p spaces with $p < \infty$. As an instance of this, verify that Minkowski's inequality and Hölder's inequality hold for all $p \in [1, \infty]$ (the dual exponent of 1 is ∞ and vice versa). Also see Exercise 20

4. Further exercises.

Exercise 14. Show that $L^p(\mathbb{R}^d) \not\subset L^q(\mathbb{R}^d)$ for all $p, q \in [1, \infty]$ with $p \neq q$.

Exercise 15. (a) Suppose that $\mu(X) < \infty$. Then $L^p(\mu) \subset L^q(\mu)$ for all $1 \leq q \leq p \leq \infty$.
(b) Suppose that there exists $c > 0$ such that if $\mu(E) < c$, then $\mu(E) = 0$. Then $L^p(\mu) \subset L^q(\mu)$ for all $1 \leq p \leq q \leq \infty$.

Exercise 16. Let $1 \leq p < r < q \leq \infty$ and assume that $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Prove that

$$\|f\|_r \leq \|f\|_p^\theta \|f\|_q^{1-\theta}$$

where $\theta \in (0, 1)$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. In particular, $f \in L^r(\mathbb{R}^d)$.

Exercise 17. Suppose $f \in L^p(\mathbb{R}^d)$ for every $1 \leq p < \infty$. Does it follow that $f \in L^\infty(\mathbb{R}^d)$? (Proof or counterexample.)

Exercise 18. Let $p \in [1, \infty]$. Give an example for a function $f \in L^p(\mathbb{R}^d)$ such that $f \notin L^q(\mathbb{R}^d)$ for all $q \neq p$.

Exercise 19. Let $X = \bigcap_{p \in [1,\infty]} L^p(\mathbb{R}^d)$. Show that X is a dense subspace of $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty)$.

Exercise 20. Suppose that f is a bounded measurable function on $\mathbb{R}^d$ which is supported on a set of finite measure. Prove that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

Exercise 21. Construct an integrable function $f : \mathbb{R} \to [0, \infty)$ such that for every g that is equal to f almost everywhere, g is unbounded on every open interval.

Exercise 22. Let $f \in L^p(\mu)$ for some $1 \leq p < \infty$. Prove that for every $\lambda > 0$ we have

$$\mu(\{x \in X : f(x) > \lambda\})^{1/p} \leq \lambda^{-1} \|f\|_p.$$

Exercise 23. Let $1 \leq p < \infty$. Let $(f_n)_n$ be a sequence in $L^p(\mu)$ that converges to some $f \in L^p(\mu)$ (in L^p-norm).
(a) Prove that for every $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

(b) Construct a sequence $(f_n)_n$ convergent in $L^p(\mathbb{R}^d)$ such that $\lim_{n\to\infty} f_n(x)$ does not exist for a.e. $x \in \mathbb{R}^d$.

Exercise 24. Prove that continuous functions with compact support are dense in $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$.