

Math 629, Spring 2020 (Roos) – Challenge problems.

March 12.

If you complete any of these problems and hand your solution in to me in writing (e-mail is okay) at the latest on the last day of class, you will receive extra credit. The amount of extra credit may be quite significant, but will be based on correctness and quality of exposition (readability). You are only allowed to hand in at most one attempt for each problem.

Please note that this is supposed to be an individual challenge. That means you have to work on it and complete the solution by yourself. You are not allowed to share your solution with classmates. You may find these problems quite difficult. Don't attempt to solve them if you are struggling with the material.

1. Define $A \subset (0, 1)$ to be the set of all $x \in (0, 1)$ such that there exists $c > 0$ such that for all integers k and all non-negative integers $j \geq 0$ we have

$$|x - k2^{-j}| \geq c2^{-j}.$$

Prove that A has Lebesgue measure zero.

2. Let $E \subset \mathbb{R}^d$. Take note of the following *ad hoc* definition: we call E a *nullset* if for every $\varepsilon > 0$ there exist cubes $(I_k)_{k=1,2,\dots}$ such that $\sum_{k=1}^{\infty} |I_k| \leq \varepsilon$ and $\bigcup_{k=1}^{\infty} I_k \supset E$. (Of course we know from class that E is a nullset if and only if it has Lebesgue outer measure zero.)

Let $E \subset \mathbb{R}^2$ and define for each $y \in \mathbb{R}$ the *slice*

$$E^y = \{x \in \mathbb{R} : (x, y) \in E\}.$$

The following statement follows from Fubini's theorem:

$E \subset \mathbb{R}^2$ is a nullset if and only if $E^y \subset \mathbb{R}$ is a nullset for almost every $y \in \mathbb{R}$.

Prove this statement directly from the definition of a nullset; that is, *without* using Fubini's theorem or other tools from measure and integration theory (but it is recommended you use the measure and integration theory to find inspiration; one approach is to 'unpack' all the definitions).

Academic Integrity Notice: As explained, you are not allowed to copy a solution or part of a solution from others (peers, internet, literature, ..). If you attempt to get credit for a stolen solution and I notice it, there will be serious consequences (at the very least you will be reported and probably fail the class).