Math 629, Spring 2020 (Roos) – Homework 1.
Due Monday, February 10.

(Problems with an asterisk (*) are optional; problems with two asterisks (**) are optional and may be more challenging.)

1. Define two outer measures on $\mathbb{R}$ as follows: for $E \subset \mathbb{R}$,

$$\mu_*^o(E) = \inf\Big\{ \sum_{i=1}^\infty |I_i| \,:\, \bigcup_{i=1}^\infty I_i \supset E,\, I_i \,\text{open intervals}\Big\},$$

$$\mu_*^c(E) = \inf\Big\{ \sum_{i=1}^\infty |I_i| \,:\, \bigcup_{i=1}^\infty I_i \supset E,\, I_i \,\text{closed intervals}\Big\}.$$

Prove that $\mu_*^o(E) = \mu_*^c(E)$ for every $E \subset \mathbb{R}$.

2. For every bounded set $E \subset \mathbb{R}$ define

$$j_*(E) = \inf\Big\{ \sum_{i=1}^N |I_i| \,:\, N \in \mathbb{N},\, \bigcup_{i=1}^N I_i \supset E,\, I_i \,\text{open intervals}\Big\}.$$

(i) Prove that $j_*(E) = j_*(\overline{E})$ for every $E \subset \mathbb{R}$, where $\overline{E} \subset \mathbb{R}$ denotes the closure of E.

(ii) Determine a set $E \subset \mathbb{R}$ such that $j_*(E) > \mu_*^o(E)$ (with μ_*^o defined as above).

3*. A *box* in $\mathbb{R}^n$ is a Cartesian product of n bounded intervals (each may be open, closed or half-open). If B is a box, then we denote its volume by $|B|$, defined as the product of the lengths of constituent intervals. Define $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^n)$ as the collection of all subsets of $\mathbb{R}^n$ that arise by taking a finite number of unions and intersections of boxes.

(i) Prove that for every $A \in \mathcal{A}$ there exist finitely many pairwise disjoint boxes $B_1, \dots, B_N$ such that

$$A = \bigcup_{i=1}^N B_i.$$

We define $\mu_0(A) = \sum_{i=1}^N |B_i|$.

(ii) Prove that $\mu_0(A)$ is well–defined (that is, independent of the chosen box decomposition of A).

(iii) Prove that

$$\mu_0\big(\bigcup_{i=1}^\infty A_i\big) = \sum_{i=1}^\infty \mu_0(A_i)$$

holds whenever $A_i \in \mathcal{A}$ for all $i = 1, \dots$ are pairwise disjoint and in addition $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$.

Remark. If we make $\mathcal{A}$ larger by also allowing set complements, (i)–(iii) continue to hold as long as we also allow unbounded boxes.

4*. In this exercise we will prove that there does not exist a measure on $\mathbb{R}$ that is defined on all subsets of $\mathbb{R}$ and is consistent with our intuitive notion of 'length'.
Define an equivalence relation on $\mathbb{R}$ by saying that $x \sim y$ if $x - y$ is rational. Let $V \subset [0,1]$ be such that V contains exactly one element of each equivalence class with respect to $\sim$. In other words, for every $r \in \mathbb{R}$ there exists a unique $v \in V$ such that $v - r$ is rational. (We will take the existence of such a V for granted; it follows from the axiom of choice.)

(i) Prove that

$$[0,1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (V + q) \subset [-1,2]$$

(Here $V + q = \{v + q \,:\, v \in V\}$.)

(ii) Prove that $(V + q) \cap (V + q') = \emptyset$ if q, q' are two distinct rational numbers.

(iii) Show that there does not exist a measure $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ (defined on all subsets of $\mathbb{R}$) satisfying $\mu(E + x) = \mu(E)$ for all $E \subset \mathbb{R}$, $x \in \mathbb{R}$ and $0 < \mu([0,1]) < \infty$.

5.** Let $U \subset \mathbb{R}^n$ be a bounded open set.
(i) Prove that there exist countably many closed cubes $Q_1, Q_2, \ldots$ with pairwise disjoint interiors such that

$$U = \bigcup_{i=1}^{\infty} Q_i$$

(ii) Show that there exist constants $c, C > 0$ only depending on n such that the cubes $Q_1, Q_2, \ldots$ in (i) can be chosen such that

$$c \cdot \ell(Q_i) \leq \operatorname{dist}(Q_i, \mathbb{R}^n \setminus U) \leq C \cdot \ell(Q_i)$$

Here, by a *cube* Q we mean a Cartesian product of bounded intervals with equal length $\ell(Q)$. Moreover, $\operatorname{dist}(A, B) = \inf\{|a - b| \,:\, a \in A, b \in B\}$.